Advanced topics on inflation

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Four epochs of the history of the Universe

$H \equiv \frac{\dot{a}}{a}$ where $a(t)$ is a scale factor of an isotropic homogeneous spatially flat universe (a Friedmann-Lemaître-Robertson-Walker background):

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) + small\ perturbations$$

The history of the Universe in one line: four main epochs

$$? \longrightarrow DS \Longrightarrow FLRWRD \Longrightarrow FLRWMD \Longrightarrow \overline{DS} \longrightarrow ?$$

Geometry

$$|\dot{H}| << H^2 \Longrightarrow H = \frac{1}{2t} \Longrightarrow H = \frac{2}{3t} \Longrightarrow |\dot{H}| << H^2$$

Physics

$$p \approx -\rho \Longrightarrow p = \rho/3 \Longrightarrow p \ll \rho \Longrightarrow p \approx -\rho$$

Duration in terms of the number of e-folds $\ln(a_{fin}/a_{in})$

> 60 ~ 55 7.5 0.5

Inflation

The inflationary scenario is based on the two cornerstone independent ideas (hypothesis):

1. Existence of inflation (or, quasi-de Sitter stage) – a stage of accelerated, close to exponential expansion of our Universe in the past preceding the hot Big Bang with decelerated, power-law expansion.

2. The origin of all inhomogeneities in the present Universe is the effect of gravitational creation of particles and field fluctuations during inflation from the adiabatic vacuum (no-particle) state for Fourier modes covering all observable range of scales (and possibly somewhat beyond).

NB The latter effect requires breaking of the weak and null energy conditions for matter inhomogeneities.

Outcome of inflation

In the super-Hubble regime in the coordinate representation:

$$ds^2 = dt^2 - a^2(t)(\delta_{lm} + h_{lm})dx^l dx^m, \quad l, m = 1, 2, 3$$

$$h_{lm} = 2\zeta(\mathbf{r})\delta_{lm} + \sum_{a=1}^{2} g^{(a)}(\mathbf{r})\, e_{lm}^{(a)}$$

$$e_l^{l(a)} = 0, \; g^{(a)}_{,l}\, e_m^{l(a)} = 0, \; e_{lm}^{(a)}\, e^{lm(a)} = 1$$

ζ describes primordial scalar perturbations, g – primordial tensor perturbations (primordial gravitational waves (GW)).

The most important quantities:

$$n_s(k) - 1 \equiv \frac{d \ln P_\zeta(k)}{d \ln k}, \quad r(k) \equiv \frac{P_g}{P_\zeta}$$

In fact, metric perturbations h_{lm} are quantum (operators in the Heisenberg representation) and remain quantum up to the present time. But, after omitting of a very small part, decaying with time, they become commuting and, thus, equivalent to classical (c-number) stochastic quantities with the Gaussian statistics (up to small terms quadratic in $\zeta, \ g$).

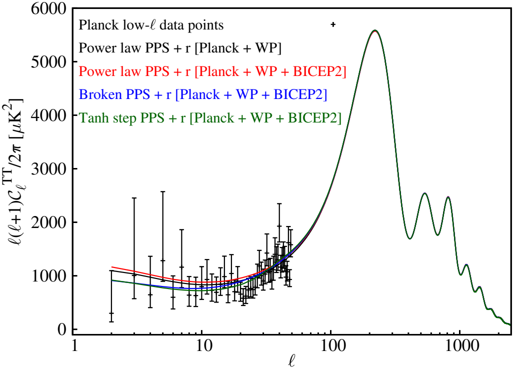
In particular:

$$\hat{\zeta}_k = \zeta_k \, i(\hat{a}_{\mathbf{k}} - \hat{a}^\dagger_{\mathbf{k}}) + \mathcal{O}\left((\hat{a}_{\mathbf{k}} - \hat{a}^\dagger_{\mathbf{k}})^2\right) + ... + \mathcal{O}(10^{-100})(\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}) + ,,,$$

The last term is time dependent, it is affected by physical decoherence and may become larger, but not as large as the second term.

Remaining quantum coherence: deterministic correlation between $\mathbf{k}$ and $-\mathbf{k}$ modes - shows itself in the appearance of acoustic oscillations (primordial oscillations in case of GW).

CMB temperature anisotropy multipoles



Present status of inflation

Now we have numbers: P. A. R. Ade et al., arXiv:1502.01589

The primordial spectrum of scalar perturbations has been measured and its deviation from the flat spectrum $n_s = 1$ in the first order in $|n_s - 1| \sim N^{-1}$ has been discovered (using the multipole range $\ell > 40$):

$$< \zeta^2(\mathbf{r}) >= \int \frac{P_\zeta(k)}{k}\, dk, \ \ P_\zeta(k) = \left(2.21^{+0.07}_{-0.08}\right) 10^{-9} \left(\frac{k}{k_0}\right)^{n_s - 1}$$

$$k_0 = 0.05\mathrm{Mpc}^{-1}, \ \ n_s - 1 = -0.035 \pm 0.005$$

Two fundamental observational constants of cosmology in addition to the three known ones (baryon-to-photon ratio, baryon-to-matter density and the cosmological constant). Existing inflationary models can predict (and predicted, in fact) one of them, namely $n_s - 1$, relating it finally to $N_H = \ln \frac{k_B T_\gamma}{\hbar H_0} \approx 67.2$ (note that $(1 - n_s)N_H \sim 2$).

From "proving" inflation to using it as a tool

Present status of inflation: transition from "proving" it in general and testing some of its simplest models to applying the inflationary paradigm to investigate particle physics at super-high energies and the actual history of the Universe in the remote past using real observational data on $n_s(k) - 1$ and $r(k)$.

The reconstruction approach – determining curvature and inflaton potential from observational data – a kind of inverse dynamical problem.

The most important quantities:
1) for classical gravity – $H, \dot{H}$
2) for super-high energy particle physics – m^2_{infl}.

Physical scales related to inflation

"Naive" estimate where I use the reduced Planck mass $\tilde{M}_{Pl} = (8\pi G)^{-1}$.

I. Curvature scale

$$H \sim \sqrt{P_\zeta}\tilde{M}_{Pl} \sim 10^{14}\mathrm{GeV}$$

II. Inflaton mass scale

$$|m_{infl}| \sim H\sqrt{|1-n_s|} \sim 10^{13}\mathrm{GeV}$$

New range of mass scales significantly less than the GUT scale.

Dynamical origin of scalar perturbations

Local duration of inflation in terms of $N_{tot} = \ln\left(\frac{a(t_{fin})}{a(t_{in})}\right)$ is different in different points of space: $N_{tot} = N_{tot}(\mathbf{r})$. Then

$$\zeta(\mathbf{r}) = \delta N_{tot}(\mathbf{r})$$

Correct generalization to the non-linear case: the space-time metric after the end of inflation at super-Hubble scales

$$ds^2 = dt^2 - a^2(t)e^{2N_{tot}(\mathbf{r})}(dx^2 + dy^2 + dz^2)$$

First derived in A. A. Starobinsky, Phys. Lett. B **117**, 175 (1982) in the case of one-field inflation.

CMB temperature anisotropy

$$T_\gamma = (2.72548 \pm 0.00057)\mathrm{K}$$

$$\Delta T(\theta, \phi) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\theta, \phi)$$

$$< a_{\ell m} a_{\ell' m'} >= C_\ell \delta_{\ell \ell'} \delta_{m m'}$$

Theory: averaging over realizations.
Observations: averaging over the sky for a fixed ℓ.

For scalar perturbations, generated mainly at the last scattering surface (the surface or recombination) at $z_{LSS} \approx 1090$ (the Sachs-Wolfe, Silk and Doppler effects), but also after it (the integrated Sachs-Wolfe effect).
For GW – only the ISW works.

Visualizing small differences in the number of e-folds

For $\ell \lesssim 50$, neglecting the Silk and Doppler effects, as well as the ISW effect due the presence of dark energy,

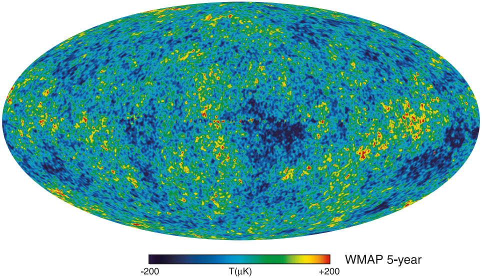
$$\frac{\Delta T(\theta,\phi)}{T_\gamma} = -\frac{1}{5}\zeta(r_{LSS},\theta,\phi) = -\frac{1}{5}\delta N_{tot}(r_{LSS},\theta,\phi)$$

For $n_s = 1$,

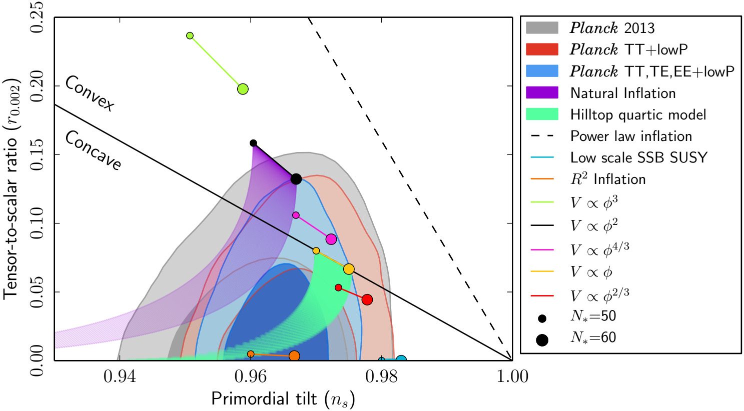
$$\ell(\ell+1)C_{\ell,s} = \frac{2\pi}{25}P_\zeta$$

For $\frac{\Delta T}{T} \sim 10^{-5}$, $\delta N \sim 5 \times 10^{-5}$, and for $H \sim 10^{14}\,\mathrm{GeV}$, $\delta t \sim 5 t_{Pl}$!

Planck time intervals are seen by the naked eye!


WMAP 5-year
-200
T(μK)
+200

Direct approach: comparison with simple smooth models



The latest BICEP2/Keck Array/Planck upper limit: $r < 0.07$ at 95% c.f. (P. A. R. Ade et al., arXiv:1510.09217).

The simplest models producing the observed scalar slope

$$f(R) = R + \frac{R^2}{6M^2}$$

$$M = 2.6 \times 10^{-6} \left(\frac{55}{N}\right) M_{Pl} \approx 3.2 \times 10^{13}\,\mathrm{GeV}$$

$$n_s - 1 = -\frac{2}{N} \approx -0.036, \quad r = \frac{12}{N^2} \approx 0.004$$

$$H_{dS}(N = 55) = 1.4 \times 10^{14}\,\mathrm{GeV}$$

The same prediction from a scalar field model with $V(\phi) = \frac{\lambda\phi^4}{4}$ at large ϕ and strong non-minimal coupling to gravity $\xi R\phi^2$ with $\xi < 0,\ |\xi| \gg 1$, including the Brout-Englert-Higgs inflationary model.

The Lagrangian density for the simplest 1-parametric model:

$$\mathcal{L} = \frac{R}{16\pi G} + \frac{N^2}{288\pi^2 P_\zeta(k)} R^2 = \frac{R}{16\pi G} + 5 \times 10^8 R^2$$

The quantum effect of creation of particles and field fluctuations works twice in this model:
a) at super-Hubble scales during inflation, to generate space-time metric fluctuations;
b) at small scales after inflation, to provide scalaron decay into pairs of matter particles and antiparticles (AS, 1980, 1981).
The most effective decay channel: into minimally coupled scalars with $m \ll M$. Then the formula

$$\frac{1}{\sqrt{-g}} \frac{d}{dt}(\sqrt{-g}\, n_s) = \frac{R^2}{576\pi}$$

can be used (Ya. B. Zeldovich and A. A. Starobinsky, JETP Lett. 26, 252 (1977)). Scalaron decay into graviton pairs is suppressed (A. A. Starobinsky, JETP Lett. 34, 438 (1981)).

Possible microscopic origins of this model.

1. The specific case of the fourth order gravity in 4D

$$\mathcal{L} = \frac{R}{16\pi G} + AR^2 + BC_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + (\text{small rad. corr.})$$

for which $A \gg 1, \; A \gg |B|$. Approximate scale (dilaton) invariance and absence of ghosts in the curvature regime $A^{-2} \ll (RR)/M_P^4 \ll B^{-2}$.

2. Another, completely different way: a non-minimally coupled scalar field with a large negative coupling ξ ($\xi_{conf} = \frac{1}{6}$):

$$L = \frac{R}{16\pi G} - \frac{\xi R\phi^2}{2} + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi), \;\; \xi < 0, \;\; |\xi| \gg 1 \; .$$

In this limit, the Higgs-like scalar tree level potential $V(\phi) = \frac{\lambda(\phi^2-\phi_0^2)^2}{4}$ just produces $f(R) = \frac{1}{16\pi G}\left(R + \frac{R^2}{6M^2}\right)$ with $M^2 = \lambda/24\pi\xi^2 G$ and $\phi^2 = |\xi|R/\lambda$ (plus small corrections $\propto |\xi|^{-1}$).

Quantization of perturbations

Quantization with the adiabatic vacuum initial condition (in the tensor case, omitting the polarization tensor):

$$\hat{\phi} = (2\pi)^{-3/2} \int \left[\hat{a}_{\mathbf{k}}\, \phi_k(\eta)\, e^{-i\mathbf{kr}} + \hat{a}_{\mathbf{k}}^{\dagger}\, \phi_k^*\, e^{i\mathbf{kr}}\right] d^3k$$

where ϕ stands for ζ, g^a correspondingly and ϕ_k satisfies the equation

$$\frac{1}{f}(f\phi_k)'' + \left(k^2 - \frac{f''}{f}\right)\phi_k = 0, \quad \eta = \int \frac{dt}{a(t)}$$

For GW: $f = a$, for scalar perturbations in scalar field driven inflation in GR: $f = \frac{a\dot{\phi}}{H}$ where, in turn, the background scalar field satisfies the equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0$$

How the two basic hypothesis of the inflationary paradigm work.

I. Inflationary background: $t = \infty$ corresponds to $\eta = 0$ and $H(\eta) \equiv \frac{a'}{a^2}$ is bounded and slowly decreasing in this limit, so that $\frac{f''}{f} \sim \frac{2}{\eta^2}$. Then

$$\eta \to -0: \quad \phi_k(\eta) \to \phi(k) = const, \quad P(k) = \frac{k^3 \phi^2(k)}{2\pi^2}$$

II. Adiabatic vacuum initial condition:

$$\eta \to -\infty: \quad \phi_k(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}}$$

The reconstruction problem: determine $H(t)$ and $V(\phi)$ from $P(k)$.

It can be solved analytically in the slow-roll approximation which is satisfied in all viable inflationary models in the first approximation: smooth reconstruction.

Inflation in GR

Inflation in GR with a minimally coupled scalar field with some potential.

In the absence of spatial curvature and other matter:

$$H^2 = \frac{\kappa^2}{3}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right)$$

$$\dot{H} = -\frac{\kappa^2}{2}\dot{\phi}^2$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

where $\kappa^2 = 8\pi G$ ($\hbar = c = 1$).

Reduction to the first order equation

It can be reduced to the first order Hamilton-Jacobi-like equation for $H(\phi)$. From the equation for $\dot{H}$, $\frac{dH}{d\phi} = -\frac{\kappa^2}{2}\dot{\phi}$. Inserting this into the equation for H^2, we get

$$\frac{2}{3\kappa^2}\left(\frac{dH}{d\phi}\right)^2 = H^2 - \frac{\kappa^2}{3}V(\phi)$$

Time dependence is determined using the relation

$$t = -\frac{\kappa^2}{2}\int\left(\frac{dH}{d\phi}\right)^{-1} d\phi$$

However, during oscillations of ϕ, $H(\phi)$ acquires non-analytic behaviour of the type $const + \mathcal{O}(|\phi - \phi_1|^{3/2})$ at the points where $\dot{\phi} = 0$, and then the correct matching with another solution is needed.

Inflationary slow-roll dynamics

Slow-roll occurs if: $|\ddot{\phi}| \ll H|\dot{\phi}|, \;\; \dot{\phi}^2 \ll V$, and then $|\dot{H}| \ll H^2$.

Necessary conditions: $|V'| \ll \kappa V, \;\; |V''| \ll \kappa^2 V$. Then

$$H^2 \approx \frac{\kappa^2 V}{3}, \;\; \dot{\phi} \approx -\frac{V'}{3H}, \;\; N \equiv \ln \frac{a_f}{a} \approx \kappa^2 \int_{\phi_f}^{\phi} \frac{V}{V'}\, d\phi$$

First obtained in A. A. Starobinsky, Sov. Astron. Lett. 4, 82 (1978) in the $V = \frac{m^2\phi^2}{2}$ case and for a bouncing model.

Spectral predictions of the one-field inflationary scenario in GR

Scalar (adiabatic) perturbations:

$$P_\zeta(k) = \frac{H_k^4}{4\pi^2\dot{\phi}^2} = \frac{GH_k^4}{\pi|\dot{H}|_k} = \frac{128\pi G^3 V_k^3}{3V_k'^2}$$

where the index k means that the quantity is taken at the moment $t = t_k$ of the Hubble radius crossing during inflation for each spatial Fourier mode $k = a(t_k)H(t_k)$. Through this relation, the number of e-folds from the end of inflation back in time $N(t)$ transforms to $N(k) = \ln\frac{k_f}{k}$ where $k_f = a(t_f)H(t_f)$, t_f denotes the end of inflation.
The spectral slope

$$n_s(k) - 1 \equiv \frac{d\ln P_\zeta(k)}{d\ln k} = \frac{1}{\kappa^2}\left(2\frac{V_k''}{V_k} - 3\left(\frac{V_k'}{V_k}\right)^2\right)$$

is small by modulus – confirmed by observations!

Tensor perturbations (A. A. Starobinsky, JETP Lett. 50, 844 (1979)):

$$P_g(k) = \frac{16GH_k^2}{\pi}; \quad n_g(k) \equiv \frac{d\ln P_g(k)}{d\ln k} = -\frac{1}{\kappa^2}\left(\frac{V_k'}{V_k}\right)^2$$

The consistency relation:

$$r(k) \equiv \frac{P_g}{P_\zeta} = \frac{16|\dot{H}_k|}{H_k^2} = 8|n_g(k)|$$

Tensor perturbations are always suppressed by at least the factor $\sim 8/N(k)$ compared to scalar ones. For the present Hubble scale, $N(k_H) = (50-60)$. Typically, $|n_g| \leq |n_s - 1|$, so $r \leq 8(1-n_s) \sim 0.3$ – confirmed by observations!

Inverse reconstruction of an inflationary model in GR

In the slow-roll approximation:

$$\frac{V^3}{V'^2} = CP_\zeta(k(t(\phi))), \quad C = \frac{12\pi^2}{\kappa^6}$$

Changing variables for ϕ to $N(\phi)$ and integrating, we get:

$$\frac{1}{V(N)} = -\frac{\kappa^4}{12\pi^2}\int \frac{dN}{P_\zeta(N)}$$

$$\kappa\phi = \int dN\sqrt{\frac{d\ln V}{dN}}$$

Here, $N \gg 1$ stands both for $\ln(k_f/k)$ at the present time and the number of e-folds back in time from the end of inflation. First derived in H. M. Hodges and G. R. Blumenthal, Phys. Rev. D 42, 3329 (1990).

Minimal "scale-free" reconstruction

Minimal inflationary model reconstruction avoiding introduction of any new physical scale both during and after inflation and producing the best fit to the Planck data.

Assumption: the numerical coincidence between $2/N_H \sim 0.04$ and $1 - n_s$ is not accidental but happens for all $1 \ll N \lesssim 60$: $P_\zeta = P_0 N^2$. Then:

$$V = V_0 \frac{N}{N + N_0} = V_0 \tanh^2 \frac{\kappa\phi}{2\sqrt{N_0}}$$

$$r = \frac{8 N_0}{N(N + N_0)}$$

$r \sim 0.003$ for $N_0 \sim 1$. From the upper limit on r:

$$N_0 < \frac{0.07 N^2}{8 - 0.07 N}$$

$N_0 < 57$ for $N = 57$.

Another example: $P_\zeta = P_0 N^{3/2}$.

$$V(\phi) = V_0 \frac{\phi^2 + 2\phi\phi_0}{(\phi + \phi_0)^2}$$

Not bounded from below (of course, in the region where the slow-roll approximation is not valid anymore). Crosses zero linearly.

More generally, the two "aesthetic" assumptions – "no-scale" scalar power spectrum and $V \propto \phi^{2n}$, $n = 1, 2...$ at the minimum of the potential – lead to $P_\zeta = P_0 N^{n+1}$, $n_s - 1 = -\frac{n+1}{N}$ unambiguously. From this, only $n = 1$ is permitted by observations. Still an additional parameter appears due to tensor power spectrum – no preferred one-parameter model (if the $V(\phi) \propto \phi^2$ model is excluded).

Inflation in $f(R)$ gravity

The simplest model of modified gravity (= geometrical dark energy) considered as a phenomenological macroscopic theory in the fully non-linear regime and non-perturbative regime.

$$S = \frac{1}{16\pi G}\int f(R)\sqrt{-g}\, d^4x + S_m$$

$$f(R) = R + F(R), \quad R \equiv R^{\mu}_{\mu}$$

Here $f''(R)$ is not identically zero. Usual matter described by the action S_m is minimally coupled to gravity.

Vacuum one-loop corrections depending on R only (not on its derivatives) are assumed to be included into $f(R)$. The normalization point: at laboratory values of R where the scalaron mass (see below) $m_s \approx \mathrm{const}$.

Metric variation is assumed everywhere. Palatini variation leads to a different theory with a different number of degrees of freedom.

Field equations

$$\frac{1}{8\pi G}\left(R_\mu^\nu - \frac{1}{2}\,\delta_\mu^\nu R\right) = -\left(T^\nu_{\mu\,(vis)} + T^\nu_{\mu\,(DM)} + T^\nu_{\mu\,(DE)}\right) \ ,$$

where $G = G_0 = const$ is the Newton gravitational constant measured in laboratory and the effective energy-momentum tensor of DE is

$$8\pi G T^\nu_{\mu\,(DE)} = F'(R)\,R_\mu^\nu - \frac{1}{2}\,F(R)\delta_\mu^\nu + \left(\nabla_\mu\nabla^\nu - \delta_\mu^\nu\nabla_\gamma\nabla^\gamma\right)F'(R)\,.$$

Because of the need to describe DE, de Sitter solutions in the absence of matter are of special interest. They are given by the roots $R = R_{dS}$ of the algebraic equation

$$Rf'(R) = 2f(R)\ .$$

The special role of $f(R) \propto R^2$ gravity: admits de Sitter solutions with any curvature.

Degrees of freedom

I. In quantum language: particle content.
1. Graviton – spin 2, massless, transverse traceless.
2. Scalaron – spin 0, massive, mass - R-dependent:
$m_s^2(R) = \frac{1}{3f''(R)}$ in the WKB-regime.

II. Equivalently, in classical language: number of free functions of spatial coordinates at an initial Cauchy hypersurface.
Six, instead of four for GR – two additional functions describe massive scalar waves.

Thus, $f(R)$ gravity is a non-perturbative generalization of GR.
It is equivalent to scalar-tensor gravity with $\omega_{BD} = 0$ if $f''(R) \neq 0$.

Transformation to the Einstein frame and back

In the Einstein frame, free particles of usual matter do not follow geodesics and atomic clocks do not measure proper time.

From the Jordan (physical) frame to the Einstein one:

$$g^E_{\mu\nu} = f' g^J_{\mu\nu}, \quad \kappa\phi = \sqrt{\frac{3}{2}} \ln f', \quad V(\phi) = \frac{f'R - f}{2\kappa^2 f'^2}$$

where $\kappa^2 = 8\pi G$.

Inverse transformation:

$$R = \left(\sqrt{6}\kappa \frac{dV(\phi)}{d\phi} + 4\kappa^2 V(\phi) \right) \exp\left(\sqrt{\frac{2}{3}} \kappa\phi \right)$$

$$f(R) = \left(\sqrt{6}\kappa \frac{dV(\phi)}{d\phi} + 2\kappa^2 V(\phi) \right) \exp\left(2\sqrt{\frac{2}{3}} \kappa\phi \right)$$

$V(\phi)$ should be at least C^1.

Why R-dependence only?

For almost all other local geometric invariants – $R_{\mu\nu}R^{\mu\nu}$, $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$, $R_{,\mu}R^{,\mu}$ etc. (where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor) – ghosts appear if the theory is taken in full, in the non-perturbative regime.

The only known exception: $f(R,G)$ with $f_{RR}f_{GG} - f_{RG}^2 = 0$, where $G = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R\mu\,\nu R^{\mu\nu} + R^2$ is the Gauss-Bonnet invariant, does not possess ghosts but has other problems.

For $f_{RR}f_{GG} - f_{RG}^2 \neq 0$, a ghost was found (A. De Felice and T. Tanaka, Progr. Theor. Phys. **124**, 503 (2010)).

Reduction to the first order equation

In the absence of spatial curvature and $\rho_m = 0$, it is always possible to reduce these equations to a first order one using either the transformation to the Einstein frame and the Hamilton-Jacobi-like equation for a minimally coupled scalar field in a spatially flat FLRW metric, or by directly transforming the 0-0 equation to the equation for $R(H)$:

$$\frac{dR}{dH} = \frac{(R - 6H^2)f'(R) - f(R)}{H(R - 12H^2)f''(R)}$$

Analogues of large-field (chaotic) inflation: $F(R) \approx R^2 A(R)$ for $R \to \infty$ with $A(R)$ being a slowly varying function of R, namely

$$|A'(R)| \ll \frac{A(R)}{R}\ , \ |A''(R)| \ll \frac{A(R)}{R^2}\ .$$

Analogues of small-field (new) inflation, $R \approx R_1$:

$$F'(R_1) = \frac{2F(R_1)}{R_1}\ , \ F''(R_1) \approx \frac{2F(R_1)}{R_1^2}\ .$$

Thus, all inflationary models in $f(R)$ gravity are close to the simplest one over some range of R.

Perturbation spectra in slow-roll $f(R)$ inflationary models

Let $f(R) = R^2\,A(R)$. In the slow-roll approximation $|\ddot{R}| \ll H|\dot{R}|$:

$$P_\zeta(k) = \frac{\kappa^2 A_k}{64\pi^2 A_k'^2 R_k^2}, \quad P_g(k) = \frac{\kappa^2}{12 A_k \pi^2}$$

$$N(k) = -\frac{3}{2}\int_{R_f}^{R_k} dR\,\frac{A}{A' R^2}$$

where the index k means that the quantity is taken at the moment $t = t_k$ of the Hubble radius crossing during inflation for each spatial Fourier mode $k = a(t_k)H(t_k)$.

Smooth reconstruction of inflation in $f(R)$ gravity

$$f(R) = R^2 A(R)$$

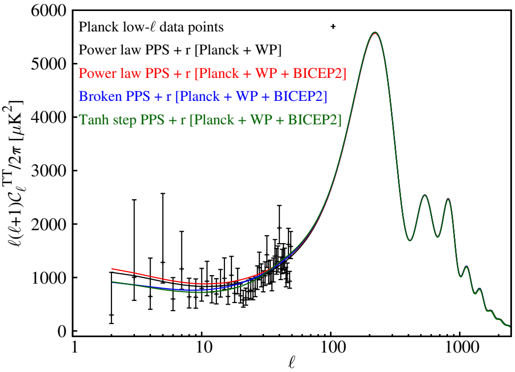
$$A = const - \frac{\kappa^2}{96\pi^2} \int \frac{dN}{P_\zeta(N)}$$

$$\ln R = const + \int dN \sqrt{-\frac{2\, d \ln A}{3\, dN}}$$

Here, the additional assumptions that $P_\zeta \propto N^\beta$ and that the resulting $f(R)$ can be analytically continued to the region of small R without introducing a new scale, and it has the linear (Einstein) behaviour there, leads to $\beta = 2$ and the $R + R^2$ inflationary model with $r = \frac{12}{N^2} = 3(n_s - 1)^2$ unambiguously.

Next step: small features in the power spectrum

1) A $\sim 10\%$ depression for $20 \lesssim \ell \lesssim 40$.
2) An upward wiggle at $\ell \approx 40$ (the Archeops feature) and a downward one at $\ell \approx 22$.



Local features in an inflaton potential

Some new physics beyond one slow-rolling inflaton may show itself through these features.

The simplest models with two additional parameters which can describe such behaviour (to some extent) are based on the exactly soluble model considered in A. A. Starobinsky, JETP Lett. **55**, 489 (1992): an inflaton potential with a sudden change of its first derivative.
For comparison of some elaborated class of such models with the CMB TT data, see
D. K. Hazra, A. Shafieloo, G. F. Smoot and A. A. Starobinsky, JCAP **1408**, 048 (2014).

Recent comparison of the model

$$V(\phi) = \theta(\phi_0 - \phi) V_1 \left(1 - \exp(-\alpha\kappa\phi)\right) +$$

$$\theta(\phi - \phi_0) V_2 \left(1 - \exp(-\alpha\kappa(\phi - \phi_1)\right)$$

where

$$V_1 \left(1 - \exp(-\alpha\kappa\phi_0)\right) = V_2 \left(1 - \exp(-\alpha\kappa(\phi_0 - \phi_1)\right)$$

with the Planck2015 data on both CMB temperature anisotropy and E-mode polarization (D. K. Hazra, A. Shafieloo, G. F. Smoot and A. A. Starobinsky, JCAP **1606**, 007 (2016); arXiv:1605.0210) provides ~ 12 improvement in χ^2 fit to the data compared to best-fit models with smooth inflaton potentials, partly due to the better description of wiggles at both $\ell \approx 40$ and $\ell \approx 22$.

Generality of inflation

Theorem. In these models, there exists an open set of classical solutions with a non-zero measure in the space of initial conditions at curvatures much exceeding those during inflation which have a metastable inflationary stage with a given number of e-folds.

For the GR inflationary model this follows from the generic late-time asymptotic solution for GR with a cosmological constant found in A. A. Starobinsky, JETP Lett. 37, 55 (1983). For the $R + R^2$ model, this was proved in A. A. Starobinsky and H.-J. Schmidt, Class. Quant. 4, 695 (1987).

Generic late-time asymptote of classical solutions of GR with a cosmological constant Λ both without and with hydrodynamic matter (also called the Fefferman-Graham expansion):

$$ds^2 = dt^2 - \gamma_{ik} dx^i dx^k$$

$$\gamma_{ik} = e^{2H_0 t} a_{ik} + b_{ik} + e^{-H_0 t} c_{ik} + \ldots$$

where $H_0^2 = \Lambda/3$ and the matrices a_{ik}, b_{ik}, c_{ik} are functions of spatial coordinates. a_{ik} contains two independent physical functions (after 3 spatial rotations and 1 shift in time + spatial dilatation) and can be made unimodular, in particular. b_{ik} is unambiguously defined through the 3-D Ricci tensor constructed from a_{ik}. c_{ik} contains a number of arbitrary physical functions (two - in the vacuum case, or with radiation).

Generic initial conditions near a curvature singularity in these models: anisotropic and inhomogeneous (though quasi-homogeneous locally).

1. Modified gravity models (the $R + R^2$ and Higgs ones).
Structure of the singularity at $t \to 0$:

$$ds^2 = dt^2 - \sum_{i=1}^{3} |t|^{2p_i} a_l^{(i)} a_m^{(i)} dx^l dx^m, \ 0 < s < 3/2, \ u = s(2-s)$$

where $s = \sum_i p_i, \ u = \sum_i p_i^2$ and $a_l^{(i)}, \ p_i$ are functions of $\mathbf{r}$. Here $R \propto |t|^{1-s} \to \infty$ (for $1 < s < 3/2$, otherwise it approaches a constant) and $R^2 \ll R_{\alpha\beta}R^{\alpha\beta}$. No infinite number of BKL oscillations.

2. GR model with a very flat potential.
A similar behaviour but with $s = 1, \ u < 1$ and with negligible potential.

In both cases, spatial gradients may become important for some period before the beginning of inflation.

Thus, the appearance of an inflating patch does not require that all parts of this patch should be causally connected at the beginning of inflation. What is needed instead in classical (modified) gravity, is:
1) the existence of a sufficiently large compact expanding ($K > 0$) region of space with the Riemann curvature much exceeding that during the end of inflation ($\sim M^2$) ;
2) the average value $< R >$ over this region positive and much exceeding $\sim M^2$, too;
3) the average spatial curvature over the region is either negative, or not too positive.

On the other hand, causal connection is certainly needed to have a "graceful exit" from inflation, i.e. to have practically the same amount of the total number of e-folds during inflation N_{tot} in some sub-domain of this inflating patch.

Weakly non-local UV-complete gravity models

$R + R^2$ gravity interacting with quantum matter fields is renormalizable in the scalar sector and can be even asymptotically free. However, in the tensor sector a ghost appears due to the squared Weyl term.

To avoid it, a subclass of weakly non-local (quasi-polynomial) UV-complete quadratic in curvature generalizations of gravity is considered which do not have ghosts and are super-renormalizable (or even finite). Their action is:

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R + R\mathcal{F}(\Box)R + C_{\mu\nu\rho\sigma}\mathcal{F}_{\rm C}(\Box)C^{\mu\nu\rho\sigma}\right]$$

where $zF(z)$ and $zF_{\rm C}(z)$ are exponentials of entire functions up to constants.

R^2 inflation as a particular solution of non-local gravity

A. S. Koshelev, L. Modesto, L. Rachwal and A. A. Starobinsky, Occurrence of exact R^2 inflation in non-local UV-complete gravity, JHEP **1611**, 067 (2016); arXiv:1604.03127

For the $R + R^2$ model, $\Box R = M^2 R$. Thus, its solutions are also particular solutions of this non-local gravity if, in symbolic notation,

$$\mathcal{F}(M^2) = \frac{M_P^2}{12M^2}, \quad \mathcal{F}'(M^2) = 0$$

Spectrum of scalar perturbations: the same is in the local $R + R^2$ model. For proving it, the fact that these perturbations are conformally flat ($\Phi + \Psi = 0$) at the inflationary stage in the leading slow-roll approximation plays a crucial role.

Tensor perturbations are different. The absence of the tensor ghost requires:

$$1 + \frac{12M^2}{M_P^2}\left(\bar{\Box} - \frac{\bar{R}}{3}\right)\mathcal{F}_{\mathrm{C}}\left(\bar{\Box} + \frac{\bar{R}}{3}\right) = \exp(2\omega(\bar{\Box}))$$

where $\omega(z)$ is some entire function and the bar means a background solution. As a result:

$$r = \frac{12}{N^2}\exp\left(2\omega\left(\frac{\bar{R}}{6}\right)\right)$$

Conclusions

- The typical inflationary predictions that $|n_s - 1|$ is small and of the order of N_H^{-1}, and that r does not exceed $\sim 8(1 - n_s)$ are confirmed. Typical consequences following without assuming additional small parameters: $H_{55} \sim 10^{14}\,\mathrm{GeV},\ \ m_{infl} \sim 10^{13}\,\mathrm{GeV}$.
- Though the Einstein gravity plus a minimally coupled inflaton remains sufficient for description of inflation with existing observational data, modified (in particular, scalar-tensor or $f(R)$) gravity can do it as well.
- From the scalar power spectrum $P_\zeta(k)$, it is possible to reconstruct an inflationary model both in the Einstein and $f(R)$ gravity up to one arbitrary physical constant of integration.

- In the Einstein gravity, the simplest inflationary models permitted by observational data are two-parametric, no preferred quantitative prediction for r, apart from its parametric dependence on $n_s - 1$, namely, $\sim (n_s - 1)^2$ or larger.
- In the $f(R)$ gravity, the simplest model is one-parametric and has the preferred value $r = \frac{12}{N^2} = 3(n_s - 1)^2$.
- Thus, it has sense to search for primordial GW from inflation at the level $r > 10^{-3}$!

- Investigation of small local features in the primordial power spectrum (bispectrum, etc.) of scalar perturbations can provide "tomography" of inflation and may lead to discovery of other particles or quasiparticles more massive than inflaton.
- Inflation is generic in the $R + R^2$ inflationary model and close ones. Thus, its beginning does not require causal connection of all parts of an inflating patch of space-time (similar to spacelike singularities). However, graceful exit from inflation requires approximately the same number of e-folds during it for a sufficiently large compact set of geodesics. To achieve this, causal connection inside this set is necessary (though still may appear insufficient).
- Solutions of the $R + R^2$ inflationary model can also be particular solutions of some non-local UV-complete modifications of gravity without ghosts.