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Yang–Mills Equations in 4-Dimensional Conformally Connected Manifolds

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Abstract—In this paper on the simplest examples of compact 4-dimensional conformally connected manifolds (real quadrics in a 5-dimensional projective space) we show that the only invariant, which is quadratic with respect to the curvature Φ of the connectivity, is the Yang–Mills functional $\int |\mathrm{tr}\,(*\Phi \wedge \Phi)|$. We do not know whether the 4-form $|\mathrm{tr}\,(*\Phi \wedge \Phi)|$ is invariant in *any* 4-dimensional conformally connected manifolds.

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1. Real quadrics in a 5-dimensional projective space P_5. A *real quadric* Q in a 5-dimensional projective space P_5 in projective coordinates obeys the equation

$$(x,x) = \eta_{11}\,(x_1)^2 + \eta_{22}\,(x_2)^2 + \eta_{33}\,(x_3)^2 + \eta_{44}\,(x_4)^2 + 2x_0x_5 = 0, \tag{1}$$

where $\eta_{ii} = \pm 1$, and the symbol (x,x) means the scalar product of points of P_5 with respect to the polar bilinear form

$$(x,y) = \eta_{11}x_1y_1 + \eta_{22}x_2y_2 + \eta_{33}x_3y_3 + \eta_{44}x_4y_4 + x_0y_5 + x_5y_0 = 0.$$

The *signature of quadric* (1) is a pair of numbers (p,q), where $(p-1)$ is the number of positive coefficients and $(q-1)$ is the number of negative ones amongst η_{ii}. The quadric Q represents a connected compact orientable 4-dimensional manifold. The invariance group of the quadric Q is a special orthogonal group $SO\,(p,q)$. It consists of matrices S of the sixth order with real elements that satisfy the identity

$$S^{\top} g S = g, \tag{2}$$

where $S^{\top}$ means the transposed matrix,

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \eta & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \eta = \begin{pmatrix} \eta_{11} & 0 & 0 & 0 \\ 0 & \eta_{22} & 0 & 0 \\ 0 & 0 & \eta_{33} & 0 \\ 0 & 0 & 0 & \eta_{44} \end{pmatrix}.$$

It is well-known that the group $SO\,(p,q)$ is simple. With $p+q=6$ it depends on fifteen parameters. Evidently, equality (2) yields $S^{\top}\,(-g)\,S = -g$, whence it follows that the groups $SO\,(p,q)$ and $SO\,(q,p)$ are isomorphic. Therefore we can restrict ourselves to three types of signatures: $(5,1)$, $(4,2)$, and

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