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A. D. Alexandrov's Problem for Non-Positively Curved Spaces in the Sense of Busemann

P. D. Andreev[1*]

[1] *Pomor State University, pr. Lomonosova 4, Arkhangelsk, 163002 Russia*

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Abstract—This paper is the last of a series devoted to the solution of Alexandrov's problem for non-positively curved spaces. Here we study non-positively curved spaces in the sense of Busemann. We prove that isometries of a geodesically complete connected at infinity proper Busemann space X are characterized as follows: If a bijection $f : X \to X$ and its inverse f^{-1} preserve distance 1, then f is an isometry.

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1. INTRODUCTION

This paper is the last of a series [1–3] devoted to the solution of Alexandrov's problem for non-positively curved spaces. In the previous papers, we studied Alexandrov non-positively curved spaces. Here we study non-positively curved spaces in the sense of Busemann (see the definition in [4] and [5–7]).

The main result of the paper is the following theorem.

Theorem 1.1. *Let (X, d_X) and (Y, d_Y) be locally compact geodesically complete connected at infinity spaces non-positively curved in the sense of Busemann, and let $f : X \to Y$ be a bijection. Then the following statements are equivalent*:

1) *For $x, y \in X$, the equality $d_X(x, y) = 1$ holds if and only if the equality $d_Y(f(x), f(y)) = 1$ holds.*

2) *For $x, y \in X$, the inequality $d_X(x, y) \le 1$ holds if and only if the inequality $d_Y(f(x), f(y)) \le 1$ holds.*

3) *For $x, y \in X$, the inequality $d_X(x, y) < 1$ holds if and only if the inequality $d_Y(f(x), f(y)) < 1$ holds.*

4) *f is an isometry of (X, d_X) onto (Y, d_Y).*

In particular, in the class of non-positively curved spaces in the sense of Busemann, we give a partial answer to the following question related to the well-known A. D. Alexandrov's problem: Under which conditions, imposed on a given metric space (X, d), every bijective mapping $f : X \to X$ preserving together with its inverse f^{-1} a fixed distance $r > 0$, for example, $r = 1$, is an isometry?

It is obvious that statements 1)–3) follow from statement 4). Theorem 1.1 has the following equivalent formulation.

Theorem 1.2. *Let, on a set X, two metrics d_1 and d_2 be given such that each of the spaces (X, d_i), $i = 1, 2$, satisfies the conditions of Theorem* 1.1. *Then the following statements are equivalent*:

*E-mail: pdandreev@mail.ru.