ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2008, Vol. 52, No. 8, pp. 12–19. © Allerton Press, Inc., 2008.
Original Russian Text © M.S. Bichegkuev, 2008, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2008, No. 8, pp. 16–24.

Bounded Solutions of Difference Inclusions

M. S. Bichegkuev[1*]

[1] *North Ossetian State University,*
ul. Vatutina 46, Vladikavkaz, 362025 Republic of North Ossetia-Alania, Russia

Received June 20, 2006

Abstract—We establish the necessary and sufficient conditions for the uniqueness of a solution to a difference inclusion in the space of bilateral vector sequences. The proof of the main result is based on the spectral theory of linear relations (multivalued linear operators).

DOI: 10.3103/S1066369X08080021

Key words and phrases: *linear relation, spectrum of a linear relation, difference inclusion, multiplication operator.*

Assume that X is a complex Banach space and A is a closed linear relation on X, i.e., A is a linear closed subspace of $X \times X$. Let Ax, where $x \in D(A)$ (see Section 1), stand for the set of vectors $\{y \in X : (x, y) \in A\}$. Consider Banach spaces $l_p = l_p(\mathbb{Z}, X)$, $1 \leqslant p \leqslant \infty$, of bilateral sequences $x : \mathbb{Z} \to X$ of vectors from X with the norm

$$\|x\|_p = \left(\sum_{n \in \mathbb{Z}} \|x(n)\|^p \right)^{\frac{1}{p}}, \ \ 1 \leqslant p < \infty,$$
$$\|x\|_\infty = \sup_{n \in \mathbb{Z}} |x(n)|, \ \ p = \infty.$$

The main results are connected with the questions of existence and uniqueness of solutions in l_p to the following difference inclusion:

$$x(n) \in Ax(n-1) + f(n), \ \ n \in \mathbb{Z}, \tag{1}$$

where $f \in l_p$, $1 \leqslant p \leqslant \infty$. Here we understand a solution to this inclusion as a sequence $x \in l_p$ which satisfies inclusions (1).

Difference inclusions naturally arise in the study of solutions in l_p of difference equations in the form

$$Bx(n) = Cx(n-1) + g(n), \ \ n \in \mathbb{Z}, \tag{2}$$

where $g \in l_p$ and $B, C : X \to Y$ are linear bounded operators which act from X to a Banach space Y (the operator B has a nonzero kernel), as well as in the study of linear differential inclusions [1] and a special class of integral operators [2]. We reduce the question on the existence of solutions to Eq. (2) in l_p to the question on the existence of a proper difference inclusion in the form (1) (see Section 1 for the summary of the main notions of the theory of linear relations).

The following theorem is the main result of this paper.

Theorem 1. *The difference inclusion* (1) *has a unique solution* $x \in l_p$, $1 \leqslant p \leqslant \infty$, *for any sequence* f *from* l_p, *if and only if the spectrum* $\sigma(A)$ *of the relation* A *is such that*

$$\sigma(A) \cap \mathbb{T} = \emptyset, \tag{3}$$

where $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ *is the unit circle in* $\mathbb{C}$.

We obtain applications for the difference equation (2) and formulas for solutions to inclusion (1). See paper [3] for an analog of Theorem 1 for a closed linear operator A.

*E-mail: bichegkuev@yandex.ru.