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A Spline Method for the Solution of Integral Equations of the Third Kind

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We consider the following linear integral equation of the third kind (ETK):

$$(Ax)(t) \equiv (Ux)(t) + (Kx)(t) = y(t), \tag{1}$$

where $(Ux)(t) \equiv x(t)t^{p_1}(1-t)^{p_2}\prod_{j=1}^{q}(t-t_j)^{m_j}$, $(Kx)(t) \equiv \int_0^1 K(t,s)x(s)ds$, $t \in I \equiv [0,1]$, $p_1, p_2 \in R^+$, $t_j \in (0,1)$, $m_j \in N$ $(j=\overline{1,q})$; K and y are known continuous functions with certain properties of pointwise "smoothness", and x is the desired function. These equations are connected with several problems of the theory of elasticity, scattering of particles, neutron transfer (see, e. g., [1] and references therein; [2]). As a rule, the intrinsic classes of solutions of the ETK are special spaces of distributions. The equations under consideration allow explicit solution in rare special cases. Therefore both the theory and the applications require approximative solution methods with the corresponding theoretical support. Several relevant results are obtained in papers [3], [4]. In [3] we propose and prove special direct solution methods for ETK (1) in the space of distributions $D\{p_1, p_2; \overline{m}, \overline{\tau}\}$. Paper [4] is dedicated to the construction of the complete solvability theory for the equations under consideration in a certain space of distributions $V\{p_1, p_2; \overline{m}, \overline{\tau}\}$.

In this paper, following the works [3]–[6], we propose and substantiate in the sense of [7] (Chap. 1) one special direct method for the approximate solution of ETK (1) in the space $V\{p_1, p_2; \overline{m}, \overline{\tau}\}$. We prove that the constructed method is optimal by the order of exactness on a certain class Φ generated by the class H_ω^r among all projection solution methods for ETK (1).

1. The space of trial functions. Let $C \equiv C(I)$ be the space of continuous on I functions with the customary max-norm and $m \in N$. According to [8], we denote by $C_{t_0}^{\{m\}} \equiv C\{m; t_0\}$ the class of functions $g \in C$ such that at a point $t_0 \in (0,1)$ they have the Taylor derivative $g^{\{m\}}(t_0)$ of the order m.

Let $t_1, t_2, \dots, t_q$ be arbitrarily fixed pairwise distinct points of the interval $(0,1)$. We associate each point t_j with a certain number $m_j \in N$ $(j=\overline{1,q})$. Let us introduce the vector space

$$C\{\overline{m}; \overline{\tau}\} \equiv C_{\overline{\tau}}^{\{\overline{m}\}}(I) \equiv \bigcap_{j=1}^{q} C\{m_j; t_j\},$$

where $\overline{m} \equiv (m_1, m_2, ..., m_q)$, $\overline{\tau} \equiv (t_1, t_2, ..., t_q)$ are finite sets of the corresponding values. Let $p_1 \in R^+$. Denote by $C\{p_1; 0\}$ the space of functions $g \in C$ with the right Taylor derivatives $g^{\{i\}}(0)$ $(i = \overline{1, [p_1]})$ at the point $t = 0$; in the case $p_1 \neq [p_1]$ ($[\,\cdot\,]$ stands for the integer part) the following finite limit exists:

$$\lim_{t\to 0+}\{[g(t) - \sum_{i=0}^{[p_1]} g^{\{i\}}(0)t^i/i!]t^{-p_1}\}.$$

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