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The Haagerup Problem on Subadditive Weights on W^*-Algebras. II

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Abstract—In 1975 U. Haagerup has posed the following question: *Whether every normal subadditive weight on a W^*-algebra is σ-weakly lower semicontinuous*? In 2011 the author has positively answered this question in the particular case of abelian W^*-algebras and has presented a general form of normal subadditive weights on these algebras. Here we positively answer this question in the case of finite-dimensional W^*-algebras. As a corollary, we give a positive answer for subadditive weights with some natural additional condition on atomic W^*-algebras. We also obtain the general form of such normal subadditive weights and norms for wide class of normed solid spaces on atomic W^*-algebras.

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Introduction. In the present paper we solve the problem posed by U. Haagerup [1] in the case of finite-dimensional W^*-algebras. As a corollary, we obtain a solution of the problem for subadditive weights with natural additional condition on atomic W^*-algebras. We also present the general form of such normal subadditive weights and norms for wide class of normed solid spaces on atomic W^*-algebras. For abelian W^*-algebras the solution to the problem posed by U. Haagerup was presented in [2].

1. Normal subadditive weights on W^*-algebra. A set B is called a *directed set*, if, firstly, the set B is partially ordered, i.e., we have a relation $\alpha \leq \beta$ for some pairs of elements $\alpha, \beta \in B$, which is reflexive ($\alpha \leq \alpha$), transitive ($\alpha \leq \beta$ and $\beta \leq \gamma$ imply $\alpha \leq \gamma$), and antisymmetric ($\alpha \leq \beta$ and $\beta \leq \alpha$ imply $\beta = \alpha$), and, secondly, there exists $\gamma \in B$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$ for every pair $\alpha, \beta \in B$. *A net* is an indexed by the directed set B collection of elements from some variety X.

Let $\mathcal{M}$ be a W^*-algebra with a predual space $\mathcal{M}_*$, and let $\mathcal{M}^+$ and $\mathcal{M}_*^+$ be the cones of nonnegative elements of $\mathcal{M}$ and $\mathcal{M}_*$, respectively. The *σ-weak convergence* is the convergence in the $*$-weak topology on $\mathcal{M}$ defined by the isomorphism $\mathcal{M} \simeq (\mathcal{M}_*)^*$.

Definition 1 ([1]). A mapping $\varphi : \mathcal{M}^+ \to [0, +\infty]$ is called a subadditive weight on W^*-algebra $\mathcal{M}$ if the following conditions hold:

(i) $x \leq y \Rightarrow \varphi(x) \leq \varphi(y)$ for all $x, y \in \mathcal{M}^+$,

(ii) $\varphi(x + y) \leq \varphi(x) + \varphi(y)$ for all $x, y \in \mathcal{M}^+$,

(iii) $\varphi(\lambda x) = \lambda\varphi(x)$ for all $x \in \mathcal{M}^+$, $\lambda \geq 0$ (here $0 \cdot (+\infty) \equiv 0$).

Definition 2 ([1]). A subadditive weight on $\mathcal{M}$ is said to be *normal* if $\varphi(\sup\limits_{i\in J} x_i) = \sup\limits_{i\in J} \varphi(x_i)$ for each increasing uniformly bounded net $\{x_i\}_{i\in J} \subset \mathcal{M}^+$, and it is called *$\sigma$-weakly lower semicontinuous* if $\varphi(x) \leq \liminf\limits_{i\in J} \varphi(x_i)$ for each net $\{x_i\}_{i\in J} \subset \mathcal{M}^+$ that σ-weakly converges to $x \in \mathcal{M}^+$.

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