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ASYMPTOTICAL STRUCTURE OF RESOLVENT OF THE UNSTABLE VOLTERRA EQUATION WITH DIFFERENCE KERNEL

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The asymptotical structure of the resolvent $R(t)$ of the equation

$$x(t) = \int_0^t K(t-s)x(s)ds + f(t) \tag{1}$$

was a subject of numerous studies (see, e.g., [1]–[8]). Let $K \in L_1$. We denote by $\widehat{U}(z) = \int\limits_0^\infty e^{-zt}U(t)dt$ the Laplace transformation of the function U. Since $\widehat{R} = \frac{\widehat{K}}{1-\widehat{K}}$, the structure of R is essentially related with zeros of the function $\widehat{K}(z) - 1$ in the right half of the complex plane. If $\widehat{K}(z)-1 \neq 0$ for $\operatorname{Re} z \geq 0$, then $R \in L_1$ by the Wiener theorem (see [1]). If in the halfplane $\operatorname{Re} z \geq 0$ $\widehat{K}(z)-1$ has a finite set of zeros λ_r of integer multiplicities m_r, then by virtue of the residue theory the behavior of R is closely connected with the quasipolynomial $Q(t) = \sum e^{\lambda_r t}P_{m_r-1}(t)$, where $P_q(t)$ is a polynomial of a degree not exceeding q.

In [2]–[5], the function R was represented in the form $R = R_0 + Q$, $R_0 \in L_1$. In [6], [7], the representation $R = R_0 + Q + Q * R_0$ or $R = R_0 + Q * R_0$ was suggested, where $R_0 \in L_1$, and

$$Q * R_0(t) = \int_0^t Q(t-s)R_0(s)ds.$$

Let us note that from the latter the representation $R = Q + R_0$ can also be obtained.

Since the function $K(z)$ is analytical for $\operatorname{Re} z > 0$, all the zeros of the function $\widehat{K}(z) - 1$ in this domain have integer multiplicities. But on the imaginary axis zeros may possess fractional multiplicities. In [8], for $K \geq 0$ and $\widehat{K}(z) - 1 = z^\beta \psi(z)$, $\psi(0) \neq 0$, $\beta \in (0,1)$, the asymptotics $\int\limits_0^t R(s)ds \sim ct^\beta$ was obtained.

In this article we clarify the structure of the resolvent in the general case of a finite set of zeros of arbitrary multiplicities. The main restriction in all cited works is the requirement $t^pK \in L_1$ for a certain integer p. If on the imaginary axis $\widehat{K}(z)-1$ has a root $i\gamma$ of a multiplicity $m > 0$, then such a requirement seems to be intrinsic, because, in this case, $\widehat{K}^{(l)}(i\gamma)$ exists, $\widehat{K}^{(l)}(i\gamma) = \lim\limits_{z\to i\gamma} \widehat{K}^{(l)}(z)$, where l is the integer part of the number m and $\widehat{K}^{(l)}(z)$ is the Laplace transformation of t^lK. Note that, if $m < p$, then m is integer due to the Taylor formula. Thus, m can be fractional only in the case $t^pK \in L_1$, $t^{p+1}K \notin L_1$; moreover, $m = p + \alpha$, $\alpha \in (0,1)$.

Everywhere in what follows $P_r(t)$ stands for a polynomial of degree not exceeding r. We put $P_{-1}(t) = 0$ and $\sum\limits_1^0 = 0$.