Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 45, No. 9, pp.1–9, 2001 *UDC 519.614*

CONDITIONAL ESTIMATES OF STABILITY IN NONSYMMETRIC PROBLEM OF EIGENVALUES

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In this article we obtain estimates of perturbation of eigenvalues of a nonsymmetric matrix A of the dimension $n \times n$. As is known (see, e. g., [1], Chap. 1, § 6), for a semisimple eigenvalue λ of the matrix A the estimate is valid

$$|\lambda - \lambda^\varepsilon| \le k(\lambda)\varepsilon + O(\varepsilon^2), \tag{0.1}$$

where $k(\lambda)$ is the *local condition number* of an eigenvalue λ ($k(\lambda)$ is the least constant which can be put in estimate (0.1); for λ corresponding to the Jordan cell of order exceeding unit we put $k(\lambda) = \infty$). Here λ^ε is an eigenvalue of the perturbed matrix $A^\varepsilon : \|A - A^\varepsilon\| \le \varepsilon$. In doing so, in a natural way we assume that among all the eigenvalues of the matrix A^ε namely λ^ε is a perturbation of the eigenvalue λ. Thus, all the eigenvalues are divided into well-conditioned and ill-conditioned (with greater $k(\lambda)$) ones. Note that an analog of the condition number can be introduced also for a semisimple eigenvalue, but here for the sake of simplicity this concept will be not discussed.

As is also well-known, in the presence of ill-conditioned points of the spectrum of the matrix A, the local well-conditionality of an eigenvalue λ does not allow us to ensure in the general case the smallness of the value $|\lambda - \lambda^\varepsilon|$, since estimate (0.1) starts to work only with $\varepsilon \le \varepsilon_0$, where ε_0 depends on the position and conditionality of the remaining eigenvalues. The value ε_0 can be practically zero, and for the level of perturbations realized in practice the dominating term in the right-hand side of estimate (0.1) turns to be $O(\varepsilon^2)$. This situation is realized in the following

Example 1. Consider the exact and perturbed matrices

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1.1 & 1 & \cdots & 0 & 0 \\ & \cdots & & \cdots & & \\ & \cdots & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1.1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1.1 \end{pmatrix}, \quad A^\varepsilon = \begin{pmatrix} 1 & \varepsilon & 0 & \cdots & 0 & 0 \\ 0 & 1.1 & 1 & \cdots & 0 & 0 \\ & \cdots & & \cdots & & \\ & \cdots & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1.1 & 1 \\ \varepsilon & 0 & 0 & \cdots & 0 & 1.1 \end{pmatrix}. \tag{0.2}$$

Here the matrix A of n-th order has the simple eigenvalue $\lambda_1 = 1$ and the eigenvalue $\lambda_2 = 1.1$, which corresponds to the Jordan cell of order $n-1$. The number λ_1 is "perfectly" conditioned: $k_1(1) = 1$. The perturbed matrix A^ε differs by the $(n, 1)$-st and $(1, 2)$-nd elements from the matrix A, as has been shown in (0.2).

The characteristic equation of the matrix A^ε has the form

$$(1 - \lambda)(1.1 - \lambda)^{n-1} - (-1)^n \varepsilon^2 = 0.$$

Supported by the Russian Foundation for Basic Research, grant no. 00-01-00325.