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STABILITY OF THE EQUATIONS WITH DELAY

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The Functional Analysis has an increasing influence on the development of new ideas and methods of investigation of equations. Such an influence requires certain revision of traditional approaches to some classical problems, in particular, to those related to differential-delay equations. The concept of these equations poorly changed since Euler, and traditions become a barrier for further investigations. A revision of the classical concepts resulted in creation of a new chapter in the Analysis — *"Theory of Functional Differential Equations"*.

We call by a *functional differential equation* the following one

$$\dot{x} = \mathcal{F}x \tag{1}$$

with the operator $\mathcal{F}$ defined on a certain set $\mathbf{D}$ of differentiable functions $x : [a, b) \to \mathbf{R}^n$. Some classes of equations of that kind attracted a quickly growing interest of both pure and applied mathematicians. The foundations of the general theory of functional differential equations were given in monographs [1], [2]. To the perspectives of further development of the studies of functional differential equations there was devoted the monograph [3] and surveys [4], [5]. Here we shall restrict ourselves to the consideration of the case where the operator $\mathcal{F}$ is *Volterra* one: For arbitrary $\tau \in (a, b)$ and a pair $x_1, x_2 \in \mathbf{D}$, the equality $(\mathcal{F}x_1)(t) = (\mathcal{F}x_2)(t)$ is fulfilled on $[a, \tau]$ if $x_1(t) = x_2(t)$ on $[a, \tau]$.

Equation (1) is a generalization of the *ordinary differential equation*

$$\dot{x}(t) = f(t, x(t)). \tag{2}$$

The generalization consists of the change of the *Nemytskiĭ* operator $(\mathcal{N}x)(t) = f(t, x(t))$ to a more general operator $\mathcal{F}$.

Many investigations of equation (2) cannot be extended to the case of equation (1). This is connected with a specificity of the differential equation, which is due to the properties of so-called "local operators", which was studied in detail by mathematicians from Perm' (see [6], [7]). Let us recall that an operator $\mathcal{F}$ is said to be *local* if the value $y(t) = (\mathcal{F}x)(t)$ at any neighborhood of a point $t = t_0$ depends only on the values $x(t)$ at the same neighborhood of the point t_0. Both the operator of differentiation $\dfrac{d}{dt}$ and the Nemytskiĭ operator $\mathcal{N}$ are local operators. Thus, the equation (2) cannot serve, for instance, in contrast to the *integral differential* equation $\dot{x}(t) = \int_a^t K(t, s, x(s))ds$ or an equation of form $\dot{x}(t) = f(t, x(\theta t))$, $t \geq 0$, where $\theta \in (0, 1)$, as a model of the process $x(t)$, whose rate of change at a given instant of time $t = t_0$ essentially depends on the state of the process $x(t)$ for $t \leq t_0$.

It is necessary to note that in last fifty years many hopes were turned to another generalization of the equation (2) — so-called "ordinary differential equation in a Banach space". Here the generalization consists of the change of a finite-dimensional space $\mathbf{R}^n$ of the values $x(t)$ of the

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