Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 47, No. 9, pp.1–3, 2003 *UDC 512.55*

CLOSEDNESS OF WEAKLY REGULAR MODULES WITH RESPECT TO DIRECT SUMS

A.N. Abyzov

In this paper, all rings are assumed to be associative and with unit. A module M is called weakly regular if each its submodule which is not contained in the Jacobson radical of M contains a nonzero direct summand of M. A module M is called 1-strictly weakly regular if each its cyclic submodule which is not contained in the Jacobson radical of M is a direct summand of M. We denote by $J(R)$ and $J(M)$ the Jacobson radicals of a ring R and of a module M, respectively.

A direct summand of a weakly regular module is a weakly regular module, but the direct sum of two weakly regular modules, generally speaking, is not a weakly regular module. In this paper, we study rings over which modules retain the weak regularity property with respect to taking the direct sum of weakly regular modules of certain type.

Theorem 1. *Let R be a semiperfect ring. Then the following conditions are equivalent*:

1) $J^2(R) = 0$;
2) *every right R-module which can be represented as the direct sum of a local and a semisimple module is weakly regular*;
3) *every right R-module which can be represented as the direct sum of a weakly regular and a semisimple module is weakly regular*;
4) *every right R-module which can be represented as the direct sum of a local and a simple module is weakly regular*;
5) *every right R-module which can be represented as the direct sum of a local projective and a simple module is weakly regular*;
6) *every right R-module which can be represented as the direct sum of a projective and a semisimple module is weakly regular.*

Proof. 1) $\Rightarrow$ 2). Let $M = xR \oplus Y$, where xR is a local and Y a semisimple submodules. With no loss of generality, we can assume that $xJ \neq 0$. Show that M is a weakly regular module. Since, according to [1] (p. 78), every R-module is a sum of local submodules, it suffices to show that every local submodule not contained in $J(M)$ is a direct summand in M. Let $N = (xr + y)R$ be a local submodule which is not contained in $J(M)$ and $y \in Y$. If $xrJ = 0$, then N is a simple submodule not contained in the radical, consequently, it is a direct summand in M. In the case when $xrJ \neq 0$, the module N is local and not simple, and $Y \cap N = Y \cap J(N) \subset Y \cap J(xR \oplus Y) = J(Y) = 0$, i. e., $Y \cap N = 0$. Since $J^2(R) = 0$, it follows that in this case $xrR = xR$. Consequently, $N \oplus Y = xR \oplus Y$.

2) $\Rightarrow$ 4). The implication is obvious.

4) $\Rightarrow$ 5). The implication is obvious.

5) $\Rightarrow$ 1). Assume the contrary. Then a local projective module eR exists such that the submodule eJ^2 is nonzero. Since the module eJ is not semisimple and, according to [1] (p. 78), is a direct sum of local submodules, it follows that in eJ a local not simple submodule $M = xR$ exists. Consider the