Russian Mathematics (Iz. VUZ) — *Izvestiya VUZ. Matematika*
Vol. 49, No. 9, pp.36–48, 2005 — *UDC 517.518*

THE LIMIT BEHAVIOR OF THE GENERALIZED VALEE POUSSIN SUMS OF MULTIPLE FOURIER SERIES

A.D. Nakhman

1. Denote $G = G^N = [-\pi,\pi]^N$, $N = 1,2,\dots$; $L^p = L^p(G)$, $p \geq 1$, $L(\ln^+ L)^N = L(\ln^+ L)^N(G)$ are, correspondingly, classes of Lebesgue measurable on G functions which are 2π-periodic in each variable x_j and satisfy the condition

$$\|f\|_p = \Big(\int_{G^N} |f(\mathbf{x})|^p d\mathbf{x}\Big)^{1/p} < \infty, \quad \int_{G^N} |f(\mathbf{x})|(\ln^+ |f(\mathbf{x})|)^N d\mathbf{x} < \infty;$$

$L(G^N) = L^1(G^N)$. Let $\mathbf{Z}^N$ be the integer lattice in R^N and $\mathbf{Z}^N_+$ be the set of all its elements $\mathbf{m}$ with nonnegative components m_j,

$$\Pi_{\mathbf{m}} = \{\mathbf{l} \in \mathbf{Z}^N_+ : 0 \leq l_j \leq m_j;\ j = 1,\dots,N\},$$
$$\Pi_{\mathbf{m},\mathbf{k}} = \{\mathbf{l} \in \mathbf{Z}^N_+ : k_j \leq l_j \leq m_j;\ \mathbf{k} \in \mathbf{Z}^N_+;\ j = 1,\dots,N\}.$$

Associate each $f \in L(G^N)$ with a sequence of rectangular partial sums of its Fourier series adjoint with respect to r (for definiteness, the first r) variables

$$\widetilde{S}^r_{\mathbf{m}}(f) = \widetilde{S}^r_{\mathbf{m}}(f,\mathbf{x}) = \sum_{\mathbf{k}\in\Pi_{\mathbf{m}}} C_{\mathbf{k}}(f)\Big(\prod_{\nu=1}^{r}(-i\operatorname{sgn} k_\nu)\Big)\exp(i\mathbf{k}\cdot\mathbf{x}).$$

Here

$$C_{\mathbf{k}}(f) = \frac{1}{(2\pi)^N}\int_{G^N} f(\mathbf{t})\exp(-i\mathbf{k}\cdot\mathbf{t})d\mathbf{t}.$$

For $r = 0$ we have (by definition) partial sums $S_{\mathbf{m}}(f)$ of the ordinary (multiple) Fourier series; the symbol $\mathbf{k}\cdot\mathbf{x}$ stands for the scalar product in R^N.

Fourier sums of functions $f \in L(\ln^+ L)^{N-1}$ and their various means are sufficiently well investigated (see, e. g., [1], Chap. 17, § 2, 3; [3], [4]); the behavior of adjoint sums and their means is less studied (see, e. g., [5], Chap. 4, § 5). However, in both cases several interesting questions remain unsolved. Thus, for example, the supremum of the majority sequence for the Cesaro means is upper estimated by the maximal Hardy–Littlewood function (see, e. g., [3]). Is a similar lower bound also valid? Is it possible to improve the upper bound up to $|f(\mathbf{x})|$, proceeding from "sup" to "lim sup" for means of a rather general form? The latter question, which is interesting separately, is also connected with some applications (see [6] and Items 10, 11 below). The results of this paper (Items 7, 8, 12) give, in particular, positive answers to these questions.

In Item 2, we introduce the generalized Valee Poussin kernel and the corresponding adjoint kernel and estimate them. In Item 3, we prove a bound (in terms of $|f(x)|$) of the upper limit of the majority sequence for generalized (one-dimensional) Valee Poussin sums. In Item 7, we obtain the N-dimensional version; in addition, we describe the properties of points which satisfy these assertions. The similar result for adjoint multiple sums is obtained in Item 8. In Items 9 and 10,