

REDUCED SEMIGROUP VARIETIES

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The notions of completeness (divisibility) and reducibility play important role in the theory of abelian groups. It turns out that one can develop an approach to the study of these notions based on the theory of group varieties. Namely, it is obvious that an abelian group is complete (or, following another terminology, divisible) if and only if it does not admit homomorphisms onto non-unit groups from atoms of the lattice of varieties of abelian groups (recall that these are exhausted by the varieties of abelian groups A_p of exponent p for all prime p). Since the lattice of subvarieties of any variety of algebras is atomic, it is clear how one should define in this variety the notion of a complete algebra and, consequently, the notion of a reduced algebra as an algebra which can have only one-element complete subalgebras. These notions were introduced in [1]. In [2], we outlined problems and established some their properties.

The methodological approach to development of the structural theory of algebras indicated in [2] is in reducing of the study of arbitrary algebras from a given variety to the study of reduced and complete algebras and their extensions. Since reduced algebras are constructed with the use of extensions from algebras belonging to minimal varieties (therein lies the difference of reduced algebras from solvable ones in the sense of [3], [4], in which all such algebras belong to the same variety), this approach is most efficient for varieties with “nice” atoms (i.e., defined by nice identities and with simple structure of algebras) of the lattice of subvarieties. This criterion is fulfilled for the varieties studied in this paper: the varieties of all semigroups, groups, inverse and totally regular unary semigroups, semigroups with zero and monoids (regarded as semigroups with nullary operation). For the above mentioned varieties, we solve the problem of characterization of reduced subvarieties. This problem was formulated in [2] (Problem 2) for arbitrary algebras. Its importance is motivated by the fact that in the general case complete and reduced algebras may have very complicated structure. This suggests to “cut out” complete algebras and “cut down” types of reduced algebras. Since, by Proposition 5 of [2], the class of all reduced algebras of a variety is only a prevariety, one can try to achieve this by means of consideration of reduced varieties of algebras, i.e., varieties consisting of reduced algebras.

It should be noted that these hopes were fully justified in the case of varieties studied in the present paper. Reduced varieties are characterized here either in terms of identities or in terms of extensions of algebras from some well-known varieties (i.e., in terms of products of varieties in the sense of A.I. Maltsev [5]). It turns out that algebras belonging to these varieties are composed, with the use of extensions, from algebras belonging to a finite collection of atoms (i.e., as we say, they are finitely reduced), which are grouped in the product in some surprising way, sometimes contrary to the properties of products of varieties; this fact can be illustrated, in particular, by the results of [6] concerning monoassociative algebras.

Before passing to formulation of the main results of the paper, recall some definitions and introduce notation. Throughout the sequel the letters k, m, n, r denote, as a rule, positive integers (if it is clear from the context that some of them can be zero, we will not specify this fact), and the letter p denotes a prime number.

Let a variety \mathcal{U} of algebras of fixed signature be given. In what follows we suppose that algebras under consideration belong to \mathcal{U} . The variety of one-element algebras will be denoted by \mathcal{E} . The subclass \mathcal{K} of \mathcal{U} is said to be *nontrivial* if it is different from \mathcal{U} and \mathcal{E} . The lattice of subvarieties of a variety of algebras \mathcal{V} is denoted by $L(\mathcal{V})$.

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