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POLYNOMIAL SNAKES WITH RESPECT TO SUBSYSTEMS OF ALGEBRAIC DEGREES

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1. Notation. Statement of problem. Polynomial snakes, introduced in [1], obtained further development in works by V.K. Dzyadyk (see, e. g., [2]). Their generalization to channels with discontinuous boundaries was constructed in [3].

Let on a finite segment $[a, b]$ a certain linearly independent system of functions $\varphi_0, \dots, \varphi_n$ be given. We denote by L a set of polynomials with respect to this system. For two fixed functions g, G such that $g(x) < G(x)$ when x belongs to $[a, b]$, we put $L(g, G) = \{P \in L : g(x) \le P(x) \le G(x),\ x \in [a, b]\}$.

Let us recall that a polynomial $Z \in L(g, G)$ is termed a snake generated by the pair (g, G), or a snake of the pair (g, G), if on the segment $[a, b]$ a collection of $m = n+1$ points $x_1 < x_2 < \dots < x_m$ exists such that with k growing from 1 to m the points $M_k = (x_k, Z(x_k))$ occur in the alternating way once on the graph $\Gamma(g)$ of the function g, next on the graph $\Gamma(G)$ of the function G (starting from any of them). But if $m > n + 1$, then Z is called an excessive snake (see [3]). S. Karlin proved the assertion: If g, G are continuous on segment $[a, b]$ functions, for which a polynomial l from L exists such that $g(x) < l(x) < G(x)$ on $[a, b]$ and the system $\varphi_0, \dots, \varphi_n$ is Chebyshev, then exactly two snakes exist generated by the pair (g, G).

This result was generalized by Ye.P. Dolzhenko and Ye.A. Sevast'yanov (see [2]) to discontinuous functions g, G. The same authors established some new properties of the snakes and also their relation with approximations with a sign-sensitive weight (see [4]–[6]).

In the present article we study questions on the existence and quantity of snakes, when the system $\varphi_0, \dots, \varphi_n$ is not Chebyshev. We consider one extremal problem connected with the snakes. Results established are concretized on polynomials with respect to certain subsystems of algebraic degrees.

2. We shall assume that all functions considered below are continuous on the segment $[a, b]$. To the pair (g, G) we relate two new functions: $g_0(x) = (g(x) + G(x))/2$, $h(x) = (G(x) - g(x))/2$. Since $G(x) > g(x)$, we have $h(x) > 0$ on the segment $[a, b]$.

We introduce the functional

$$N(f) = N(f, g, G) = \max\{|f(x) - g_0(x)|/h(x) : x \in [a, b]\}.$$

As known (see [7], [8]), $N(f)$ is a continuous convex nonnegative functional and

$$L(g, G) = \{l \in L : N(l) \le 1\}. \tag{1}$$

The quantity $N(f_1 - f_2)$ will be called (g, G)-distance between f_1 and f_2. Let

$$E_n(f) = \inf_{l \in L} \max\{|f(x) - l(x)|/h(x) : x \in [a, b]\}.$$