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CONTINUITY AND CONNECTEDNESS OF METRIC δ-PROJECTION IN A UNIFORMLY CONVEX GEODESIC SPACE

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In this article the known (see [1]–[3]) results on continuity and connectedness of metric δ-projection in uniformly convex Banach space are generalized to the case of uniformly convex geodesic space. A simple consequence of such a generalization is that these results hold true for the Lobachevskiĭ spaces (including infinite-dimensional ones).

1. Preliminaries

Let (X,ρ) be a convex metric space (i. e., a metric space where the distance between two points is equal to the length of the shortest curve with its ends at these points, see [3]). A convex metric space is called a direct convex metric space (cf. [4], p. 13) if through every two distinct points one can draw a unique straight line (i. e., a geodesic curve isometric to the real straight line R with the standard metric (ibid., p. 52)). A complete convex metric space is called a geodesic space (see [5]).

We will use the following notation. In what follows $B(x,r)$ ($B[x,r]$, $S(x,r)$) means an open ball (a closed ball, a sphere) with center x and radius $r>0$; we set $xy=\rho(x,y)$, $xM=\rho(x,M)$; $[x,y]$ ((x,y)) is a closed (open) segment with ends $x,y\in X$. Consider a direct convex metric space and assume that $\omega_\lambda(x,y)$, $\lambda\in R$, is a point on the straight line passing through x, y such that a) $x\omega_\lambda(x,y)=|\lambda|xy$; b) if $\lambda\in[0,1]$, then $\omega_\lambda(x,y)\in[x,y]$; c) if $\lambda>1$, then $y\in(x,\omega_\lambda(x,y))$; d) if $\lambda<0$, then $x\in(\omega_\lambda(x,y),y)$.

A set M in a direct convex space is said to be convex if for each two distinct points x, y in M the segment $[x,y]$ lies in M.

Definition (see [6]). A direct convex metric space X is said to be uniformly convex if X satisfies the following conditions:

A) for each $\lambda\in R$ the mapping $\omega_\lambda : X\times X\to X$ is uniformly continuous on each set $B\times B$, where B is a closed ball in X;

B) each closed ball $B[p,r]$ in X is convex and has the property: If $\lim\limits_{n\to\infty} p\omega_{1/2}(x_n,y_n)=r$ for sequences $(x_n),(y_n)$ in $B[p,r]$, then

$$\lim_{n\to\infty} x_ny_n=0.$$

The uniformly convex Banach spaces and the Lobachevskiĭ spaces (including infinite-dimensional ones) are simple examples of uniformly convex geodesic spaces.

Let us recall definitions and notation from [2], [3]. Suppose that $\delta\geq 0$, $x\in X$, $M\subset X$, $M\neq\emptyset$, $t\geq 0$; $x_M^\delta=\{y\in M : xy\leq xM+\delta\}$ ($x_M=x_M^0$) is the metric δ-projection (the metric projection) of x onto M; $P : X\times 2^X\times R_+\to 2^X$, $P(x,M,\delta)=x_M^\delta$ is the operator of metric δ-projection (the operator of metric projection if $\delta\equiv 0$); $\lambda(M)=\sup\{\rho(A,B) : A\cup B=M,$

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