Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 50, No. 4, pp.56–63, 2006 *UDC 512.643*

ON SIMILARITY OF SECOND ORDER MATRICES OVER THE RING OF WHOLE NUMBERS

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It is known from the linear algebra (see, e. g., [1], p. 101) that, with a linear transformation of a linear space V over a field $\mathbf{F}$, a class of similar matrices $[\varphi]$ is associated. This class consists of all matrices of the transformation φ in terms of various bases of the space V. If $A, B \in [\varphi]$, then there exists a nondegenerate matrix S with elements from $\mathbf{F}$ such that $AS = SB$. The similarity of matrices A and B over the field of rational numbers will be denoted by $A \approx B$. For checking whether two matrices are similar over the field $\mathbf{Q}$, there is an algorithm (see, e. g., [1], p. 154–157) of reducing the characteristic matrix $A - \lambda E$ to the Smith normal diagonal form and then constructing the unique canonical matrix F (in [1], p. 181, it is called the natural normal form of a matrix A).

We transfer the notion of similarity to the ring $\mathbf{Z}$ of whole numbers. The considerable difficulties arising in this situation forced us to restrict ourselves to the study of the two-dimensional case. In this case, we construct an algorithm for checking whether two matrices A and B over $\mathbf{Z}$ are similar or not. If the characteristic polynomial of a matrix is reducible over $\mathbf{Q}$, we describe the class of similar matrices.

Definition 1. We will say that a matrix $B \in \mathbf{Z}^{n\times n}$ is similar to a matrix $A \in \mathbf{Z}^{n\times n}$ over $\mathbf{Z}$ if there exists $S \in \mathbf{Z}^{n\times n}$ such that $AS = SB$ and $\det S \in \{-1, 1\}$. This fact will be denoted by $A \sim B$. The matrix S will be called a matrix $\mathbf{Z}$-transforming A into B.

Obviously, the above introduced relation is an equivalence relation.

Consequently, the set $\mathbf{Z}^{n\times n}$ is divided into equivalence classes K_j, $j \in J$. Thus, we have

$$\mathbf{Z}^{n\times n} = \bigcup_{j\in J} K_j.$$

An integer-valued matrix A can be considered as a matrix over the residue ring $\mathbf{Z}/m$ if we put $A = (\operatorname{res}_m a_{ij})$, where m is a natural number and $\operatorname{res}_m x$ is the remainder on division of x by m. Denote by $(\mathbf{Z}/m)^*$ the set of divisors of the unit in $\mathbf{Z}/m$.

Definition 2. We will say that a matrix $B \in \mathbf{Z}^{n\times n}$ is similar to a matrix $A \in \mathbf{Z}^{n\times n}$ over $\mathbf{Z}/m$ if there exists $S \in \mathbf{Z}^{n\times n}$ such that $AS \equiv SB \pmod m$ and $\det S \in (\mathbf{Z}/m)^*$. This fact will be denoted by $A \sim_m B$.

The problem is to determine whether two given integer matrices A and B are similar, and if they are, to find a transforming matrix.

Over $\mathbf{Q}$, the criterion of similarity of matrices is the equality of the greatest common divisors (gcd) of the k-th order minors of their characteristic matrices. The generalization of the notion of gcd to the ring $\mathbf{Z}[\lambda]$ is the ideal generated by the k-th order minors of the characteristic matrix.

The research was supported in part by the Russian Foundation for Basic Research (grant 05-01-00552-a).