ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2011, Vol. 55, No. 9, pp. 76–79. © Allerton Press, Inc., 2011.
Original Russian Text © F.G. Avkhadiev, R.G. Nasibullin, and I.K. Shafigullin, 2011, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2011, No. 9, pp. 90–94.

Hardy-Type Inequalities with Power and Logarithmic Weights in Domains of the Euclidean Space

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Received February 28, 2011

Abstract—We consider Hardy-type inequalities in domains of the Euclidean space for the case when the weight depends on the distance function to the domain boundary and has power and logarithmic singularities. We prove several new inequalities with sharp constants.

DOI: 10.3103/S1066369X1109009X

Keywords and phrases: *Hardy-type inequalities, distance function to the boundary, iterations of logarithms.*

1. Introduction. Let Ω be a domain in the Euclidean space $\mathbb{R}^n$, and let $\Omega \neq \mathbb{R}^n$. Then the distance function $\delta = \delta(x, \Omega) = \operatorname{dist}(x, \partial\Omega)$, $x \in \Omega$, is well defined. Let $C_0^1(\Omega)$ be the family of continuously differentiable functions $f : \Omega \to \mathbb{R}$ with compact supports in Ω. For such functions we will consider Hardy-type inequalities in the case when the weight depends on δ and has power and logarithmic singularities. One can find the basic results on Hardy-type inequalities in [1–4], the papers present a development of the theory, new applications, and, in particular, connections with some isoperimetric problems. In the latter ten years several authors pay attention to explicit estimates of Hardy constants [15–24). For instance, the paper [17] presents the following Hardy-type inequality with the sharp constant p^p:

$$\int_\Omega \frac{|f|^p}{\delta^n} dx \le p^p \int_\Omega \frac{|\nabla f|^p}{\delta^{n-p}} \left(\ln \frac{\delta_0}{\delta}\right)^p dx.$$

The goal of this paper is to obtain more general inequalities of this type. We also study some Hardy constants for domains $\Omega = \Omega_0 \times (a, b)^j$, provided that the corresponding constants for the domain $\Omega_0 \subset \mathbb{R}^{n-j}$, $j \ge 1$, are known. To construct new weights, we use iterated logarithms [1], in proofs of new inequalities for multidimensional domains the methods from [15] and [17] are used. Notice that a basic part of our results is a collection of new one-dimensional "L^1-inequalities", special cases of which are used in the proofs of multidimensional cases. To get L^p-versions of inequalities, we use the following generalization of a result from [17].

Lemma 1. *Assume that Ω is a domain in the Euclidean space $\mathbb{R}^n$, $n \ge 1$, $w_1 = w_1(x) > 0$, $w_2 = w_2(x) \ge 0$ on Ω, and the function $w_1[w_2/w_1]^l$ is locally integrable in Ω for any $l \in [1, p]$. If $J : C_0^1(\Omega) \to \mathbb{R}$ is a functional and for any function $f \in C_0^1(\Omega)$ the following inequality:*

$$J(f) + \int_\Omega |f| w_1 dx \le c \int_\Omega |\nabla f| w_2 dx, \quad c = \text{const} > 0,$$

is valid, then for any $p \in (1, \infty)$, $l \in [1, p]$ and $f \in C_0^1(\Omega)$,

$$lJ(|f|^p) + \int_\Omega |f|^p w_1 dx \le (cp)^l \int_\Omega |f|^{p-l} |\nabla f|^l w_1^{1-l} w_2^l dx.$$

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