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On Solving the Problems of Stability by Lyapunov's Direct Method

B. S. Kalitine[1*]

[1] *Belorussian State University*
pr. Nezavisimosti 4, Minsk, 220030 Republic of Belarus

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Abstract—We consider problems of stability and instability of the trivial solution to non-autonomous systems of differential equations. We suggest new theorems of Lyapunov's direct method with the use of semi-definite auxiliary functions. The idea is based on the use of the additional function that evaluates the rate of convergence of the solutions to the set, where Lyapunov's function vanishes. We formulate theorems on the non-asymptotic stability and instability. The results are illustrated by examples, where we give a comparison with known results.

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Lyapunov's functions method ([1], pp. 77–94) is efficiently used for solving problems of motion stability in many problems of applied mathematics. For investigation of stability of the trivial solution to non-autonomous differential equations it has been developed in papers of N. G. Chetaev, K. P. Persidskii, J. Kurzweil, E. A. Barbashin, N. N. Krasovskii, V. I. Zubov, V. M. Matrosov, S. N. Vasil'ev, J. P. La-Salle, T. Yoshizawa, Z. Artstein, A. S. Andreev, S. V. Pavlikov, and others (see, e.g., [2–6]). Here an essential characteristic of solution to problems of stability, asymptotic stability, and stability in large of a trivial solution to system of equations is the requirement that an auxiliary function $V(x,t)$ must be positively defined.

In papers [5–7] for the problem about asymptotic stability (local and global) one weakened requirements on the derivative with respect to time $\dot V(x,t)$ of Lyapunov's function $V(x,t)$. One showed that instead of functions with the property of definite negativity $\dot V(x,t)$ one can use functions with the negative sign derivative, whose set of zeros $\{\dot V(x,t)=0\ \ \forall t\geq 0\}$ contains no positive semi-trajectories.

Further one obtained a generalization of theorems of the second method [8–13] for non-autonomous systems on the case, when the Lyapunov's function $V(x,t)$ is a fixed sign function, only. A distinctive characteristic of these results about stability is defined by the fact that the function $V(x,t)$ vanishes on a certain subset of the phase space, containing the origin.

We note a characteristic of use of fixed sign auxiliary functions applied to the theorem about non-asymptotic stability (NS). The presence of such functions is connected with the principle of reduction, where the question about stability with respect to the whole phase space is reduced to the search of stability conditions on a subset Y_0, on which the fixed sign function V equals zero. We obtain relative stability or Y_0-stability. It is clear that for each fixed sign Lyapunov's function $V(x)\geq 0$ with the non-positive derivative with respect to time $\dot V(x)\leq 0$ the set Y_0 is positively invariant. Therefore the stability of trivial solution necessarily implies Y_0-stability. But trivial examples show that although the property of Y_0-stability in the Lyapunov sense is a necessary condition, but it does not guarantee the NS property for the trivial solution with respect to the whole phase space.

In the general form, the problem of reduction principle has been formulated by J. S. Florio in 1968 for dynamic systems on a metric space, where, in particular, one said also about the NS problem. A year later P. Seibert gave the solution to this problem by means of topological dynamics for locally compact dynamic systems [14]. Then J. S. Florio and P. Seibert strengthened this result for dynamic systems

[*] E-mail: Kalitine@yandex.ru.