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A Weyl-Geodesic Field of Cones in a Three-Dimensional Riemannian Space II. First Integrals of Geodesics

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Abstract—We study three-dimensional Riemannian spaces and Weyl spaces such that the geodesic equations admit a first integral of second order. With respect to a special coordinate system, we find objects which completely determine connection of these spaces as well as the integrals.

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1. In the present paper we construct first integrals of second order for differential equations of geodesics in three-dimensional Riemannian spaces and Weyl spaces. In a Riemannian (Weyl) space the integral exists if and only if the connection of this space admits a geodesic field of direction cones [1], i.e., vector fields X^i $(i, j, k = \overline{1, n})$ defined by a nondegenerate symmetric tensor field a_{ij} as follows:

$$a_{ij}X^iX^j = 0 \tag{1}$$

under the assumption that the tensor field a_{ij} satisfies the equation $\nabla_{(k}a_{ij)} = M_{(k}a_{ij)}$, where M_k is a vector field and ∇_k stands for the covariant derivative (by parentheses we denote the symmetrization with respect to the indices between these parentheses).

If the connection is Riemannian and $M_k = 0$, then the relation $(a_{ij}dx^idx^j)(ds^2)^{-1} = \text{const}$ (x^i are coordinates, s is the affine parameter on geodesic) is a first quadratic integral of geodesics ([2], P. 209); if the connection under consideration is a Weyl connection determined by a main tensor g_{ij} and an additional vector ω_k ([3], P. 153), and $M_k = 2\omega_k$, then the relation $(a_{ij}dx^idx^j){:}(g_{ij}dx^idx^j)^{-1} = \text{const}$ is the first fractional quadratic integral [4] generated by the tensors a_{ij} and g_{ij}.

In [5] we considered special Weyl-geodesic fields of direction cones in a Riemannian space V_3. Indeed, we studied direction fields (1) such that the tensor a_{ij} satisfies the equation

$$\nabla_{(k}a_{ij)} = M_{(k}a_{ij)} + R_{(k}g_{ij)}, \tag{2}$$

where M_k and R_k are vector fields. We obtained a classification of these spaces using types of roots of the equation

$$|a_{ij} - \lambda g_{ij}| = 0. \tag{3}$$

The field considered above is special in the sense that the vector subspaces spanned by the eigenvectors corresponding to distinct roots of (3) have maximal dimensions, are not isotropic, and the fields of these subspaces are holonomic (see [5]).

2. Given a Riemannian connection which admits a field of cones (1), (2), we can easily construct a connection whose geodesic equations admit a quadratic first integral. To do it, we impose the requirement that $M_k = R_k = 0$.

In the same way we can construct a Weyl connection whose geodesic equations admit a fractional quadratic integral. If we assume that (2) is given in W_n, we require that $M_k^W = 2\omega_k$ and $R_k^W = 0$ (we denote by the index W vector fields in W_n). At the same time, this connection can be constructed

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