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SOLVABILITY CONDITIONS FOR A SYSTEM OF LINEAR DIOPHANTINE EQUATIONS WITH PRIME VARIABLES

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Consider a system of linear equations

$$b_i = a_{i1}p_1 + a_{i2}p_2 + \cdots + a_{im}p_m, \quad i = 1, 2, \dots, s, \tag{1}$$

where $b_1, b_2, \dots, b_s$ are natural numbers, $a_{i1}, a_{i2}, \dots, a_{im}$ are integer coefficients, and $p_1, p_2, \dots, p_m$ are unknown prime numbers. In paper [1] one studies the solvability of this system under certain additional assumptions and obtains for $m \geq 2s+1$ the asymptotic formula for a number of solutions of system (1). However, for $s < m \leq 2s$ no asymptotic formula is obtained, moreover, even the existence of solutions for an arbitrary collection of natural numbers $b_1, b_2, \dots, b_s$ is not proved.

In [2] for system (1) for $s = 2$, $m = 3$ one estimates the number of pairs (b_1, b_2), $1 \leq b_1, b_2 \leq X$, for which system (1) is unsolvable with respect to prime variables, i.e., the number of elements in the set $E_2(X) = \{(b_1, b_2) \mid 1 \leq b_1, b_2 \leq X,\ b_i = a_{i1}p_1 + a_{i2}p_2 + a_{i3}p_3,\ i = 1, 2\}$. In addition, for sufficiently large X one establishes the power bound

$$\operatorname{card} E_2(X) \leq X^{2-\varepsilon}, \tag{2}$$

where ε is an absolute, effectively calculable, positive small constant.

Putting $s = n$, $m = n+1$, from (1) we obtain the system of equations

$$b_i = a_{i1}p_1 + a_{i2}p_2 + \cdots + a_{i,n+1}p_{n+1}, \quad i = 1, 2, \dots, n. \tag{3}$$

It contains n equations with respect to $n+1$ prime variables. For convenience, denote

$$A_{i_1 i_2 \dots i_n} = \begin{bmatrix} a_{1i_1} & a_{1i_2} & \cdots & a_{1i_n} \\ a_{2i_1} & a_{2i_2} & \cdots & a_{2i_n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{ni_1} & a_{ni_2} & \cdots & a_{ni_n} \end{bmatrix}, \quad \Delta_{i_1 i_2 \dots i_n} = \det(A_{i_1 i_2 \dots i_n}).$$

Here $1 \leq i_1, i_2, \dots, i_n \leq n+1$; $\Delta_{i_1 i_2 \dots i_{n-1} b}$ is the determinant of the matrix $A_{i_1 i_2 \dots i_{n-1} b}$ obtained from that $A_{i_1 i_2 \dots i_n}$ by the replacement of the i_n-th column with that of free terms b_i of system (3). We will also use the following notation: $\Delta_i = \Delta_{12 \dots i-1, i+1, \dots n+1}$, Δ_{ik}^b is the determinant of the n-th order obtained from Δ_i by the replacement of the k-th column with that of free terms b_i of system (3).

In order to avoid the triviality and the degeneracy, let us require that the coefficients a_{ij} satisfy the conditions

$$\Delta_1 \Delta_2 \dots \Delta_{n+1} \neq 0, \quad (\Delta_1, \Delta_2, \dots, \Delta_{n+1}) = 1. \tag{4}$$

As usual, the solvability of system (1) depends on the following two conditions (see [2]; [3], Chap. 5).