

## THE LIMIT OF SYMMETRIC DIFFERENCE FOR A CAUCHY TYPE INTEGRAL

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### Introduction

We consider the integral of the Cauchy type

$$\Phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\tau)d\tau}{\tau - z} \quad (1)$$

with the continuous density  $f(\tau)$  over the smooth path  $\Gamma$ . Let  $\nu = e^{i\theta}$ ,  $\theta = \theta(t)$ , be the unit normal vector for the path  $\Gamma$  directed to the left of  $\Gamma$  with respect to the positive path-tracing. It is known (see [1], [2]) that almost everywhere on  $\Gamma$  the limit values  $\Phi^{\pm}(t) = \lim_{h \rightarrow 0+} \Phi(t \pm h\nu)$  exist and satisfy the relation  $\Phi^+(t) - \Phi^-(t) = f(t)$ ,  $t \in \Gamma$ . Generally speaking, we cannot assert that the boundary values  $\Phi^+(t)$  and  $\Phi^-(t)$  exist at any point  $t \in \Gamma$ . In this paper, we study the symmetric difference of the Cauchy type integral

$$\Delta_h \Phi(t) = \Phi(t + h\nu) - \Phi(t - h\nu), \quad t \in \Gamma, \quad (2)$$

where  $h \in \mathbb{R}_+$  ( $h$  is sufficiently small). It follows from [1], [2] that

$$\lim_{h \rightarrow 0+} \Delta_h \Phi(t) = f(t) \quad (3)$$

exists at all points  $t$  where the limit values  $\Phi^+(t)$  and  $\Phi^-(t)$  exist, i.e., almost everywhere. We will show that in reality the limit of the symmetric difference exists and equals the value of the function for any  $t \in \Gamma$  and generalize this result for integrable functions.

Let us call to our mind the definition of a Lebesgue point and reformulate it for functions defined on curves (see, for instance, [3], Chap. 10, § 1).

**Definition 1.** Let the function  $g$  be defined on the segment  $[\alpha, \beta]$ . A point  $t \in (\alpha, \beta)$  is a Lebesgue point for function  $g$  if

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} \int_t^{t+\delta} \{g(\tau) - g(t)\} d\tau = 0.$$

We reformulate this definition for function defined on a rectifiable curve  $\Gamma$  in the following way.

**Definition 2.** Let  $\tau = \tau(s)$ ,  $0 \leq s \leq l$ , be the natural parametrization of the curve  $\Gamma$ . A point  $\tau_0 = \tau(s_0)$  is called a Lebesgue point of a function  $f$  defined on  $\Gamma$  if point  $s_0$  is a Lebesgue point for the function  $g(s) = f(\tau(s))$ .

The main results of this paper are two following theorems.

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