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The Sequential Differentiation and Its Applications in the Optimal Control Problems

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Abstract—Using the sequential approach, we define a certain generalization of the operator derivative. We establish the necessary extremum condition in terms of the sequential derivative. As examples we consider the optimal control problems for systems governed by partial nonlinear differential equations of several kinds.

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The necessary conditions for an extremum and the gradient methods for minimization of functionals are connected with the differentiation. For rather difficult extremal problems the approach based on the classical derivatives of Gateaux, Frechet, etc is not always applicable. Therefore one often uses their generalizations, namely, the subdifferential calculus [1], the Clarke derivatives [2], the quasidifferentiation [3], the extended differentiation [4], etc. In [5] for the investigation of optimal control problems for systems with nonsmooth nonlinearity we introduce the notion of the sequential derivative, using the constructions of the sequential distribution theory [6]. In this paper, we propose a more efficient definition of the sequential derivative. This allows us to study from the common viewpoints qualitatively different extremal problems, whose solution in the framework of the classical differentiation theory is difficult.

1. MINIMIZATION OF SEQUENTIALLY DIFFERENTIABLE FUNCTIONALS

The very high efficiency of the theory of generalized functions is essentially caused by the closedness of the set of distributions with respect to the operation of differentiation. Together with the standard theory of S. L. Sobolev and L. Schwartz, where a distribution is understood as an object of the space adjoined to the set of infinitely differentiable finite functions, the sequential theory of distributions worked out by Ya. Mikusińskii et al. [6] has a wide application. Within this theory a generalized function means the class of equivalence of fundamental sequences of infinitely differentiable functions, and its derivative is understood as the class of equivalence of the corresponding sequences of derivatives. In [5] with the help of analogous constructions we define the sequential derivative of an operator. It is used in the optimal control problem for a system with a nonsmooth operator. Let us define a modification of the sequential derivative which turns out to be more efficient and is applicable to an essentially wider class of problems.

Consider a Banach space U with a subset W and a point $u_0 \in W$. Define a family Σ of sequences $\{I_k\}$ of functionals on U for which one can find a sequence $\{u_k^I\}$ from U such that for any k the functional I_k is Gateaux differentiable at u_k^I, the sequence $\{I_k(u)\}$ converges uniformly with respect to $u \in W$, and the limits $\{I_k(u_0)\}$ and $\{I_k(u_k^I)\}$ coincide. Introduce on Σ the equivalence relation σ, assuming that the condition $\{I_k\}\sigma\{J_k\}$ is fulfilled, if the sequence $\{I_k(u) - J_k(u)\}$ tends to zero uniformly with respect to $u \in W$. Define the factor set $T_*^S = \Sigma/\sigma$.

Remark 1. In [5] as points of differentiability of I_k we choose the value u_0; the functional I is defined on the whole space U. Therefore, the notion of the sequential derivative introduced below is more general.

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