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PLUS-OPERATORS IN NEUMANN ALGEBRA

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Non-contractive operators and plus-operators are specific for spaces with indefinite metric (J-spaces) (see [1]). In this paper, we introduce and investigate the analogs of non-contractive operators and plus-operators in Neumann algebras. For the theory of Neumann algebras we refer the reader to [2] and [3] (Chap. VI). Our method for choosing analogs of plus-vectors differs from one suggested in [4].

General properties

Let $\mathcal{M}$ be a Neumann algebra in a complex Hilbert space H with scalar product $(\cdot,\cdot)$. Let us denote by $\mathcal{Z}$ the center of $\mathcal{M}$ and by $\mathcal{Z}^{\mathrm{pr}}$ the set of all orthogonal projectors from $\mathcal{Z}$. We introduce a partial order on the set $(:= \mathcal{Z}^+)$ of all nonnegative central operators, namely $Z_1 \le Z_2$ if $(Z_1x,x) \le (Z_2x,x)\ \forall x \in H$. For any Z_1, Z_2 in $\mathcal{Z}^+$ a projector $E \in \mathcal{Z}^{\mathrm{pr}}$ exists such that $Z_1E \le Z_2E$ and $Z_2(I-E) \le Z_1(I-E)$. We set $Z_1 \wedge Z_2 := Z_1E + Z_2(I-E)$ and $Z_1 \vee Z_2 := Z_2E + Z_1(I-E)$.

The following statement is often referred to as the Vigier theorem. *Any increasing and bounded above net of bounded self-adjoint operators converges strongly to a certain self-adjoint operator. The limit operator is the least upper bound of the increasing net of operators.*

Let $P^+ and P^- := I - P^+$ be orthogonal projectors in $\mathcal{M}$ such that the central supports (see [2]) of P^+ and P^- equal to I. Let us define the canonical symmetry $J := P^+ - P^-$ and fix the indefinite metric $[x,y] := (Jx,y)$, $x,y \in H$. An operator $V \in B(H)$ is called a *plus-operator* if $[Vx,Vx] \ge 0$ for all $x \in H$ such that $[x,x] \ge 0$. In [1] a plus-operator V is called *strict* if $\inf\{[Vx,Vx] : [x,x]=1\} > 0$, otherwise V it is called *nonstrict.* Let us denote by $A^\#$ the operator conjugated to $A \in B(H)$ as respects $[\cdot,\cdot]$, i.e., $[Ax,y] = [x,A^\# y]$. It is clear that $A^\# = JA^*J$. We define the *support* of the vector $x \in H$ as the least projector Q_x in $\mathcal{Z}$ such that $x = Q_x x$.

Lemma 1. *If $[x,x] > 0$, then the greatest projector $:= Q_{+x} \in \mathcal{Z}$, $Q_{+x} \ne 0$ exists such that $[qx,x] > 0$ for any $q \in \mathcal{Z}^{\mathrm{pr}}$, $q \ne 0$, $q \le Q_{+x}$.*

Proof. Let us suppose that $[x,x] > 0$. We consider the set $\mathcal{Z}_- := \{p \in \mathcal{Z}^{\mathrm{pr}} : [px,x] < 0,\ p \le Q_x\}$. If $\mathcal{Z}_- = \emptyset$, then we set ${}_{-x}P := 0$. If $\mathcal{Z}_-$ is not empty, then we choose an arbitrary family of orthogonal in pairs projectors $\{P_i\}$ in $\mathcal{Z}_-$ which is maximal with respect to the inclusion relation. It is evident that $\sum\limits_i P_i < Q_x$. We set ${}_{-x}P := \sum P_i$. Then for any $q \in \mathcal{Z}^{\mathrm{pr}}$ such that $q \le Q_x - ({}_{-x}P)$ we get $[qx,x] \ge 0$.

Now let us consider the set $\mathcal{Z}_0 := \{p \in \mathcal{Z}^{\mathrm{pr}} : [px,x] = 0,\ p \le Q_x - {}_{-x}P\}$. Note that $0 \in \mathcal{Z}_0$. We take in $\mathcal{Z}_0$ the maximal family $\{Q_j\} \subseteq \mathcal{Z}_0$ of orthogonal in pairs projectors and set ${}_{0x}P := \sum Q_j$. It is evident that $Q := Q_x - {}_{-x}P - {}_{0x}P \ne 0$ and

$$[rx,x] > 0 \quad \forall r \in \mathcal{Z}^{\mathrm{pr}} \quad (0 \ne r \le Q). \tag{1}$$

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