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Ideal F-Norms on C^*-Algebras

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Abstract—We show that every measure of non-compactness on a W^*-algebra is an ideal F-pseudonorm. We establish a criterion of the right Fredholm property of an element with respect to a W^*-algebra. We prove that the element $-I$ realizes the maximum distance from a positive element to a subset of all isometries of a unital C^*-algebra, here I is the unit of the C^*-algebra. We also consider differences of two finite products of elements from the unit ball of a C^*-algebra and obtain an estimate of their ideal F-pseudonorms. We conclude the paper with a convergence criterion in complete ideal F-norm for two series of elements from a W^*-algebra.

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Introduction. We study ideal F-norms on C^*-algebras. We show that every measure of non-compactness on a W^*-algebra is an ideal F-pseudonorm. We establish a criterion of the right Fredholm property of an element with respect to a W^*-algebra. We prove that the minimum distance with respect to an ideal seminorm from an arbitrary element to the Hermitian (respectively, skew-Hermitian) part of a C^*-algebra is realized on the Hermitian (respectively, skew-Hermitian) part of this element. We show that the maximum of the distance with respect to an ideal F-pseudonorm from a positive element to the subset of all isometries of a unital C^*-algebra is realized on the element $-I$. We obtain an estimate of an ideal F-pseudonorm of the difference of two finite products of elements of a unit ball of a C^*-algebra. We establish a convergence criterion with respect to a complete ideal F-norm for two series consisting of elements of a W^*-algebra.

1. Definitions and notations. A C^*-algebra is a complex Banach $*$-algebra $\mathcal{A}$ such that $\|A^*A\| = \|A\|^2$ for all $A \in \mathcal{A}$. A W^*-algebra is a C^*-algebra $\mathcal{A}$, that has a predual Banach space $\mathcal{A}_*$: $\mathcal{A} \simeq (\mathcal{A}_*)^*$. For a C^*-algebra $\mathcal{A}$, let $\mathcal{A}^{\mathrm{sa}}$ and $\mathcal{A}^+$ denote its subsets of Hermitian elements and positive elements, respectively. Let $\mathcal{A}^1 = \{A \in \mathcal{A} : \|A\| \le 1\}$. If $A \in \mathcal{A}$, then $|A| = \sqrt{A^*A} \in \mathcal{A}^+$, $\Re A = (A + A^*)/2$ and $\Im A = (A - A^*)/(2i)$ lie in $\mathcal{A}^{\mathrm{sa}}$. For a unital $\mathcal{A}$, let $\mathcal{A}^{\mathrm{u}}$ and $\mathcal{A}^{\mathrm{iso}}$ denote its subsets of unitary elements ($A^*A = AA^* = I$) and isometries ($A^*A = I$), respectively.

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$, $\mathcal{B}(\mathcal{H})$ be a W^*-algebra of all linear bounded operators in $\mathcal{H}$. Any C^*-algebra can be realized as a C^*-subalgebra in $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (I. M. Gel'fand–M. A. Naimark; see [1], theorem 3.4.1).

Let $\mathcal{A}$ be a W^*-algebra. For projectors $P, Q \in \mathcal{A}$, let us write $P \sim Q$ if $P = U^*U$ and $Q = UU^*$ with some $U \in \mathcal{A}$. A projector $P \in \mathcal{A}$ is called *finite*, if $P \sim Q \le P$ implies $P = Q$; $\mathcal{A}$ is called *finite*, if the projector I is finite. Let $\mathcal{F}$ denote an ideal generated by finite, with respect to $\mathcal{A}$, projectors. A uniform closure of $\mathcal{F}$ forms an ideal $\mathcal{K}$ of compact (with respect to $\mathcal{A}$) elements. Let $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{K}$ be a canonical mapping. An element $A \in \mathcal{A}$ is called right Fredholm with respect to $\mathcal{A}$, if $\pi(A)$ is right invertible in $\mathcal{A}/\mathcal{K}$. Let us denote the set of all such elements as $\Phi^-(\mathcal{A})$.

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