ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2009, Vol. 53, No. 12, pp. 81–86. © Allerton Press, Inc., 2009.
Original Russian Text © P.I. Troshin, 2009, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2009, No. 12, pp. 95–100.

Systems of Two Iterated Functions Over Skew Field of Quaternions

P. I. Troshin[1*]

[1]*Kazan State University, ul. Kremlyovskaya 18, Kazan, 420008 Russia*

Received June 4, 2008; in final form, June 11, 2009

Abstract—Systems of linear iterated functions $f_0(z) = qz + a$, $f_1(z) = qz + b$ over the field of complex numbers have been investigated since 1985 (Barnsley and Harrington). Much attention is paid to the question of the connection of their attractors. We consider systems of iterated functions $f_0(z) = qzp + a$, $f_1(z) = qzp + b$ over the skew field of quaternions. We simplify the form of such systems and study the structure of their attractors.

DOI: 10.3103/S1066369X09120135

Key words and phrases: *system of iterated functions, attractor, quaternion.*

Introduction. Systems of iterated functions over the field of complex numbers $\mathbb{C}$ in the form

$$\begin{aligned} f_0(z) &= \widetilde{q}z + a, \\ f_1(z) &= \widetilde{q}z + b, \quad \widetilde{q}, z, a, b \in \mathbb{C}, \ |\widetilde{q}| < 1, \end{aligned} \tag{1}$$

were considered earlier by several authors in [1–3]. The connection of attractors of the mentioned systems was studied especially thoroughly (in the investigation of the Mandelbrot set). For such systems the question about the similarity is trivial: With fixed q and varying a and b the attractor of system (1) is always similar to the attractor of the system

$$\begin{aligned} f_0(z) &= \widetilde{q}z, \\ f_1(z) &= \widetilde{q}z + 1, \quad \widetilde{q}, z \in \mathbb{C}, \ |\widetilde{q}| < 1. \end{aligned} \tag{2}$$

Therefore, it suffices to study the case $a = 0$, $b = 1$.

Consider systems of iterated functions over the skew field of quaternions $\mathbb{H}$ in the form

$$\begin{aligned} f_0(z) &= qzp + a, \\ f_1(z) &= qzp + b, \qquad q, p, z, a, b \in \mathbb{H}, \ |qp| < 1. \end{aligned} \tag{3}$$

It turns out that in this case with fixed q and p and varying a and b attractors are not similar. Example 3 illustrates this fact. In this paper we show the simplest forms to which one can reduce system (3) and adduce examples of four-, three-, two-, and one-dimensional attractors of this system. We also show that in a particular case with $p = \overline{q}$ the attractor lies in a three-dimensional space and is isometric to the attractor defined by a pair of similarity mappings in $\mathbb{R}^3$. In the case $p = 1$ the attractor lies in a two-dimensional plane and is isometric to the attractor of system (1) in $\mathbb{C}$.

1. The general case: Rotations in $\mathbb{R}^4$. Consider the skew field of quaternions $\mathbb{H} = \{q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \mid a_i \in \mathbb{R}\}$ and introduce the following denotations: $\operatorname{Im}\mathbb{H} = \{q = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \mid a_i \in \mathbb{R}\}$, $|q| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$, $\overline{q} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$. Note that the product in $\mathbb{H}$ is noncommutative.

Quaternions form the four-dimensional Euclidian space. Any rotation of this space with respect to $q = 0$ can be written in the form $q \mapsto \xi q \zeta$, where $|\xi| = |\zeta| = 1$. Purely vector quaternions $\operatorname{Im}\mathbb{H}$ form the three-dimensional Euclidian space. Any rotation of the space $\operatorname{Im}\mathbb{H}$ with respect to $q = 0$ can be written in the form $q \mapsto \xi q \overline{\xi}$, where ξ is some unit quaternion (see details in [4], pp. 21–23).

*E-mail: Paul.Troshin@gmail.com.