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TOPOLOGICAL PROPERTIES OF MANIFOLDS OVER LOCAL ALGEBRA ADMITTING HOLOMORPHIC EMBEDDINGS

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1. Introduction

Let $\mathcal{A}$ be a local algebra over the field of real numbers R. In what follows we will assume that $\mathcal{A}$ is a finite-dimensional associative commutative algebra with unit. Let us denote by I the radical of $\mathcal{A}$. As a vector space over R, the algebra $\mathcal{A}$ is isomorphic to the direct sum $\mathrm{R}\oplus\mathrm{I}$. Let us denote by n the dimension of $\mathcal{A}$ over R; then we have $\mathcal{A}\approx\mathrm{R}^{\mathrm{n}}$.

Let M be a finite-dimensional manifold over $\mathcal{A}$. Recall that a real manifold M is called a manifold over $\mathcal{A}$ if M admits an atlas $\{(\mathrm{U_i},\varphi_\mathrm{i})\}_{\mathrm{i}\in\mathrm{I}}$ such that $\varphi(\mathrm{U_i})\subset\mathcal{A}^{\mathrm{m}}$ (with the same m for all i), and the coordinate transformations $\varphi_\mathrm{j}\circ\varphi_\mathrm{i}^{-1}$ are $\mathcal{A}$-holomorphic (see [1], [2]). The atlas $\{(\mathrm{U_i},\varphi_\mathrm{i})\}_{\mathrm{i}\in\mathrm{I}}$ is called an $\mathcal{A}$-atlas. Recall also that a mapping $f:(\mathrm{U}\subset\mathcal{A}^{\mathrm{p}})\to\mathcal{A}^{\mathrm{q}}$, where U is open in $\mathcal{A}^{\mathrm{p}}$, is said to be $\mathcal{A}$-holomorphic if f treated as a mapping over R lies in the class C^∞ ($\mathcal{A}$ is identified with R^{n}), and the derivative f' is $\mathcal{A}$-linear. Then m is called the dimension of M over $\mathcal{A}$. Any manifold over $\mathcal{A}$ can be treated as a real manifold. The dimension of M over R is equal to mn. In this article we study the holomorphic embeddings of a manifold M over $\mathcal{A}$ into a Cartesian power $\mathcal{A}^{\mathrm{r}}$ of $\mathcal{A}$.

2. Topological properties of manifolds over local algebra $\mathcal{A}$ holomorphically embedded into a Cartesian power $\mathcal{A}^{\mathrm{r}}$ of $\mathcal{A}$

Let M be an m-dimensional manifold over local algebra $\mathcal{A}$, and the dimension of $\mathcal{A}$ over R be n. A mapping $\mathrm{F}:\mathrm{M}\to\mathcal{A}^{\mathrm{r}}$ is called an $\mathcal{A}$-holomorphic embedding of M into $\mathcal{A}^{\mathrm{r}}$ if F is a C^∞-embedding of M which is considered as a real manifold into R^{rn} ($\mathcal{A}^{\mathrm{r}}\approx\mathrm{R}^{\mathrm{rn}}$), and F is $\mathcal{A}$-holomorphic. The latter means that M admits an $\mathcal{A}$-atlas $\{(\mathrm{U_i},\varphi_\mathrm{i})\}_{\mathrm{i}\in\mathrm{I}}$ such that the mappings $\mathrm{F}\circ\varphi_\mathrm{i}^{-1}:(\varphi_\mathrm{i}(\mathrm{U_i})\subset\mathcal{A}^{\mathrm{m}})\to\mathcal{A}^{\mathrm{r}}$ are $\mathcal{A}$-holomorphic. Let us consider the lower central series of the radical I of $\mathcal{A}$,

$$\mathrm{I}\supset\mathrm{I}^2\supset\mathrm{I}^3\supset\cdots\supset\mathrm{I}^{\mathrm{q}-1}\supset\mathrm{I}^{\mathrm{q}}\supset\mathrm{I}^{\mathrm{q}+1}=0.$$

Now let us consider the ideal $\mathrm{I}^{[\mathrm{q}/2]+1}$, where $[\mathrm{q}/2]$ stands for the integer part of q/2, and suppose that the dimension of $\mathrm{I}^{[\mathrm{q}/2]+1}$ over R is equal to k. Let us denote by $\mathrm{I}^{[\mathrm{q}/2]+1}\cdot\mathrm{T_xM}$ the subspace in the tangent space $\mathrm{T_xM}$ of M at x which consists of vectors $z=\sum_{i=1}^{\mathrm{m}}\alpha_i z_i$, where $\alpha_i\in\mathrm{I}^{[\mathrm{q}/2]+1}$, $z_i\in\mathrm{T_xM}$. The dimension of $\mathrm{I}^{[\mathrm{q}/2]+1}\cdot\mathrm{T_xM}$ over R equals mk.

Lemma. *For any holomorphic coordinate system (U,φ) on* M *and for any $u,v\in(\mathrm{I}^{[\mathrm{q}/2]+1})^{\mathrm{m}}\subset\mathrm{A}^{\mathrm{m}}$, we have $(\mathrm{F}\circ\varphi^{-1})''(u,v)=0$.*