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ON COVERING OF DISKS IN THE BIEBERBACH–EILENBERG CLASS

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The present article is devoted to the study of one subclass of the known class of the regular Bieberbach–Eilenberg functions. We use the terminology, basic concepts, and facts from [1]–[3]. As in [3]–[5], the technique of investigation represents a combination of both the method of modules in the form of module problems for two classes of curves and the symmetrization method.

In the present article we use the following notation: R stands for the class of regular in disk $U = \{z : |z| < 1\}$ functions $w = f(z) = a_1 z + a_2 z^2 + \cdots$ obeying the condition $f(z_1) \cdot f(z_2) \neq 1$ for any $z_1, z_2 \in U$, $R(\lambda)$ means a subclass in R of schlicht functions $w = f(z) = a_1 z + a_2 z^2 + \cdots$, for which the equality holds $|a_1| = \lambda$, $0 < \lambda \leq 1$; $m(D)$ stands for the conformal module of the doubly-connected domain D, i.e., the ratio of radii of boundary circles of a concentric circular ring which is conformally equivalent to the domain D; $f(U)$ means the set of values of a certain function $w = f(z)$ given in the disk U; $\frac{1}{f(U)}$ is the domain obtained from $f(U)$ by the transformation $\frac{1}{w}$; $R_\mu(\lambda)$ stands for the set of functions $w = f(z)$ from the class $R(\lambda)$, which satisfy the additional relation

$$m\{\overline{\mathbb{C}}_w \setminus [f(U) \cup \tfrac{1}{f(U)}]\} \geq \mu^2, \quad 1 \leq \mu < \tfrac{1}{\lambda}.$$

In the present article we obtain the covering theorem in the class $R_\mu(\lambda)$. The extremal function is related to a certain quadratic differential, whose critical trajectories partition the whole plane into a pair of domains, which in turn are extremal in the same module problem for two classes of curves. This module problem was considered in [3], a more general problem was solved in [6].

Let -1, 1, in, $n \geq 0$, be marked points of the plane $\mathbb{C}_w$. Let H_1 be a class of closed Jordan curves on $\overline{\mathbb{C}}' = \overline{\mathbb{C}}_w \setminus \{-1, 1, in, \infty\}$, which separate ∞ from the points -1, 1, and in. Among a countable quantity of homotopy classes of closed Jordan curves on $\overline{C}'$, which separate the points -1, 1 from the points in, ∞, we shall consider the following two classes of curves: $H_2^{(1)}$ and $H_2^{(2)}$. The class $H_2^{(1)}$ consists of curves homotopic on $\overline{\mathbb{C}}'$ to the cut along the segment $[-1, 1]$. Let $\rho > 0$ be sufficiently small, the class $H_2^{(2)}$ consisting of curves homotopic on $\overline{C}'$ to the cut along a broken straight line with the vertices at the points -1, $i(n+\rho)$, 1. Let $D^{(k)}$, $k = 1, 2$, be families of all pairs of nonoverlapping domains $\{D_1, D_2\}$, where $\infty \in D_1$, D_1 is a simply connected domain associated with the class H_1, D_2 is a doubly-connected domain associated to the class $H_2^{(k)}$, $k = 1, 2$. In this situation, we admit the case where D_2 degenerates. Let $M(D_1, \infty)$ be the reduced module of the domain D_1 with respect to the point ∞. Then $M(D_1, \infty) = \frac{1}{2\pi} \log \frac{1}{R(D_1, \infty)}$, where $R(D_1, \infty)$ is the conformal radius in the domain D_1 with respect to the point ∞, $R(D_1, \infty) = \operatorname{cap}(\partial D_1)$ is the logarithmic capacity of the boundary of the domain D_1; $M(D_2)$ is the module of doubly-connected domain D_2 with respect to the curves separating its boundary components, $M(D_2) = \frac{1}{2\pi} \log m(D_2)$. Consider the problem $m_k(n, \alpha_1, \alpha_2)$, $k = 1, 2$, on the maximum of the sum $\alpha_1^2 M(D_1, \infty) + \alpha_2^2 M(D_2)$ for all $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ such that $\alpha_1^2 + \alpha_2^2 = 1$ in the family of domains $D^{(k)}$. Such a problem is a particular case of the extremal-metric module problem for several homotopic classes of curves (see [2], Chap. 0, theorem 0.1). By concretizing the qualitative result in [6] in the general module problem, we get the