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HOLOMORPHIC MAPS OF ALMOST SEMI-KÄHLERIAN MANIFOLDS

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1. *Introduction.* The theory of harmonic maps of manifolds has been actively developed in the last few decades (see [1]; [2], pp. 171–191). Within the framework of this theory the harmonic maps between almost complex manifolds are studied, in particular the harmonic maps between Kählerian manifolds (see the references above). For example, it has long been known that each holomorphic map is harmonic (see, e. g., [3] and [2], p. 172).

In the present paper we study the harmonicity of holomorphic maps of almost semi-Kählerian manifolds, quasi-Kählerian manifolds, and nearly Kählerian manifolds.

2. *Definitions and Results.* Recall that an *almost Hermitian manifold* (M, g, J) is a $2m$-dimensional smooth manifold M endowed with almost complex structure J (a smooth section of the bundle $T^*M \otimes TM$ such that $J^2 = -\,\mathrm{id}$) and a (pseudo-)Riemannian metric g compatible with J (i. e., $g(J, J) = g$) (see [4], p. 139). Let us denote by ∇ the Levi-Civita connection on M corresponding to g. Then (M, g, J) is said to be *almost semi-Kählerian* (see, e. g., [5]) if $\nabla^* J = 0$, where the operator ∇^* is formally adjoint to ∇. The semi-Kählerian manifold (M, g, J) can be specialized as follows: this manifold is said to be *quasi-Kählerian*, *nearly Kählerian*, and *Kählerian* [6], if $(\nabla_X J)Y + (\nabla_{JX} J)JY = 0$, $(\nabla_X J)X = 0$, and $\nabla J = 0$, respectively.

Now let us consider a smooth map $f : M \to M'$, which needs not to be a diffeomorphism, where (M, g, J) and (M', g', J') are almost Hermitian manifolds. The map $f : M \to M'$ is called *holomorphic* [1] or, in other terms, *almost complex* ([4], p. 118), if the differential $f_* : TM \to TM'$ commutes with the almost complex structures J and J' of manifolds M and M', respectively, i. e.

$$f_* \circ J = J' \circ f_*. \tag{2.1}$$

For Kählerian manifolds M and M', each holomorphic map $f : M \to M'$ satisfies the Euler–Lagrange equation [1], from which it follows that f is a harmonic map,

$$\mathrm{trace}_g\, \overline{\nabla} f_* = 0, \tag{2.2}$$

where $\overline{\nabla}$ is the natural connection in the bundle $T^*M \otimes f^{-1}(TM')$ generated by the Levi-Civita connections ∇ and ∇' of M and M', respectively.

Theorem. *For an almost semi-Kählerian (nearly Kählerian, quasi-Kählerian) manifold M and a nearly Kählerian manifold M', a holomorphic map $f : M \to M'$ is harmonic.*

Corollary 1. A holomorphic immersion $f : M \to N$ of an almost Hermitian manifold (M, g, J) into a nearly Kählerian manifold (N, g', J') is harmonic if and only if (M, g, J) is a semi-Kählerian manifold.

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