Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 46, No. 12, pp.82–88, 2002 *UDC 519.10*

TYPES OF $(3n-2)$-NONINTEGER VERTICES OF THE POLYTOPE OF THREE-INDEX AXIAL PROBLEM OF CHOICE

M.K. Kravtsov, V.M. Kravtsov, and Ye.V. Lukshin

In spite of a similarity, at the first sight, between the polytope $M(2,n)$, $n \geq 2$, of the two-index assignment problem of order n and the polytope $M(3,n) = \Big\{x = \|x_{ijt}\|_n : \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} x_{ijt} = 1 \ \forall t \in N_n$, $\sum\limits_{i=1}^{n}\sum\limits_{t=1}^{n} x_{ijt} = 1 \ \forall j \in N_n$, $\sum\limits_{j=1}^{n}\sum\limits_{t=1}^{n} x_{ijt} = 1 \ \forall i \in N_n$, $x_{ijt} \geq 0 \ \forall (i,j,t) \in N_n^3\Big\}$, where $N_n = \{1,\dots,n\}$, $N_n^3 = N_n \times N_n \times N_n$, of the three-index axial problem of choice (on assignments), they possess a principal difference consisting of the fact that the polytope $M(3,n)$ is not integer, while the polytope $M(2,n)$ has only integer vertices (see [1]). Examples of noninteger vertices of the polytope $M(3,n)$ can be met in literature long ago (see [1]); however, the proper concept of an r-noninteger vertex of the polytope $M(3,n)$ was introduced recently. Let us recall (see [2]–[4]) that a vertex of the polytope $M(3,n)$ is said to be r-noninteger (an r-vertex) if it contains exactly r noninteger (fractional) components. The first basic result related to the question of existence of r-vertices of the polytope $M(3,n)$, is the theorem in [3]: for any number $r \in R_n = \{4,6,7,\dots,3n-2\}$, and only for this number, the polytope $M(3,n)$ has r-vertices. The second important result consists of the theorems on lower bounds of the number $\sigma(n,r)$ of r-vertices of the polytope $M(3,n)$ (see [2]–[4]), which made it possible to reject the hypothesis 18 in [5]. By means of these theorems and explicit formulas for $\sigma(n,r)$ for $r = 4,6,7$ (see [4], [6]) in [2], [6] bounds from below for the number of noninteger vertices of the polytope $M(p,n)$ of the p-index ($p \geq 3$) axial problem, which improve significantly the estimate given in [17], were obtained.

From the mathematical point of view, the problem of description and study of the structure of r-vertices of the polytope $M(3,n)$ for all $r \in R_n$ is of significant interest. This problem in the general case is very complex, first of all, due to a non-univalence (non-uniqueness) of the structures of the vertices of polytope, which define their type. The identification of the types of r-vertices of the polytope $M(3,n)$ is carried out by the quantity of fractional components contained in the two-dimensional sections of three-index matrices representing its vertices.

In this article we study some types of $(3n-2)$-vertices of the polytope $M(3,n)$ and investigate their properties. We assume in what follows that $n \geq 3$.

1. Properties of $(3n-2)$-vertices. The collection of elements of the matrix $x = \|x_{ijt}\|_n$ with fixed value of one index, for example, t, will be called the two-dimensional section of orientation (i,j) of the matrix x. The two-dimensional section represents a usual two-index matrix. Thus, the matrix x has two-dimensional sections of orientations (i,j), (i,t), and (j,t). An arbitrary orientation of a two-dimensional section of the matrix x will be denoted by (g,h), while a fixed two-dimensional section of this orientation by s.

The collection of elements of the matrix $x = \|x_{ijt}\|_n$ with fixed values of two indices, for example, i and j, will be called a one-dimensional section of orientation t of matrix x. In addition, the