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Infinitesimal Harmonic Transformations and Ricci Solitons on Complete Riemannian Manifolds

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Abstract—Ricci solitons were introduced by R. Hamilton as natural generalizations of Einstein metrics. A Ricci soliton on a smooth manifold M is a triple (g_0, ξ, λ), where g_0 is a complete Riemannian metric, ξ a vector field, and λ a constant such that the Ricci tensor Ric_0 of the metric g_0 satisfies the equation $-2\,\mathrm{Ric}_0 = L_\xi g_0 + 2\lambda g_0$. The following statement is one of the main results of the paper. Let (g_0, ξ, λ) be a Ricci soliton such that (M, g_0) is a complete noncompact oriented Riemannian manifold, $\int\limits_M \|\xi\| dv < \infty$, and the scalar curvature s_0 of g_0 has a constant sign on M, then (M, g_0) is an Einstein manifold.

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INTRODUCTION

The definition, local and global properties, and examples of infinitesimal harmonic transformations were considered in [1–4]. In particular, in [4], a connection was established between infinitesimal harmonic transformations and Ricci solitons. In the present paper, this result is applied to the study of infinitesimal harmonic transformations and Ricci solitons on complete Riemannian manifolds.

1. INFINITESIMAL HARMONIC TRANSFORMATIONS OF A COMPLETE RIEMANNIAN MANIFOLD

1. A vector field $\xi \in C^\infty TM$ generates in a neighborhood U of each point of a Riemannian manifold (M, g) a local one-parameter group of transformations $\varphi_t(x) = \overline{x}^k = x^k + t\xi^k$, where $x^1, x^2, \dots, x^n$ are local coordinates in U, $t \in (-\varepsilon, +\varepsilon) \subset \mathbb{R}$, and $\xi = \xi^k \partial_k$ (see [5], pp. 21–23; [6], pp. 39–41). For this reason, a vector field ξ is also called an *infinitesimal transformation* of a manifold (M, g).

If a local one-parameter group of transformations generated by a vector field ξ in a neighborhood of each point of a Riemannian manifold (M, g) consists of harmonic transformations, the vector field ξ is called an *infinitesimal harmonic transformation* of (M, g) (see [1–3]).

The following theorem holds (see [1–3]).

Theorem 1. *In order for a vector field $\xi \in C^\infty TM$ to be an infinitesimal harmonic transformation of (M, g), it is necessary and sufficient that its components satisfy the differential equations $\Delta\xi = 2\,\mathrm{Ric}^*\,\xi$ for the Hodge–De Rham Laplacian Δ and the linear operator* Ric^* *corresponding to the Ricci tensor,* $\mathrm{Ric}(\cdot,\cdot) = g(\mathrm{Ric}^*\cdot,\cdot)$.

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