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On Some Types of Boundary Points of Harmonic Functions

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Abstract—The paper deals with the Lindelöf and Fatou points of arbitrary harmonic functions defined in the unit disk. We present necessary and sufficient conditions for existence of such points on the unit circle.

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Many papers are devoted to investigation of boundary singularities of harmonic functions; in particular, we note [1–6]. In the paper we use conventional designation. Let D be the unit disk $|z| < 1$, Γ be the unit circle $|z| = 1$ and $h(\xi, \varphi)$ be the chord of the unit disk D which ends at the point $\xi = e^{i\theta} \in \Gamma$ and forms the angle φ with the radius-vector of the point, $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$. Let $\Delta(\xi, \varphi_1, \varphi_2)$ denote the subdomain of D bounded by the chords $h(\xi, \varphi_1)$ and $h(\xi, \varphi_2)$. The domain $\Delta(\xi, \varphi_1, \varphi_2)$ is usually called the Stolz angle with vertex at the point $\xi = e^{i\theta} \in \Gamma$. If we are not interested in size of the angle, we will denote it for short by $\Delta(\xi)$. We denote by $\Lambda(\xi)$ the diameter of the disk D with endpoints ξ and $-\xi$.

We can interpret D as a model of plane in Lobachevskii geometry. Denote by $\sigma(z_1, z_2)$ the non-Euclidean distance between points z_1 and z_2 in D:

$$\sigma(z_1, z_2) = \frac{1}{2} \ln \frac{1+u}{1-u}, \quad u = \left| \frac{z_1 - z_2}{1 - z_1 \overline{z}_2} \right|.$$

Consider a function $f(z)$ defined in D. For arbitrary subset S of the disk D having $\xi \in \Gamma$ as a limit point, denote by $C(f, \xi, S)$ the limit set of $f(z)$ at the point ξ with respect to S, i.e., $C(f, \xi, S) = \cap \overline{f(S \cap U(\xi))}$. Here the intersection is over all neighborhoods $U(\xi)$ of the point ξ, and the bar means the closure of set with respect to the two-point compactification $\overline{R}$ of $R = (-\infty, +\infty)$. We can imagine it as a segment after adding to the set R two symbols $-\infty$ and $+\infty$. We relate a point $\xi \in \Gamma$ to the set $K(f)$ for a function $f(z)$, defined in D, if $C(f, \xi, \Delta_1(\xi)) = C(f, \xi, \Delta_2(\xi))$ for any Stolz angles $\Delta_1(\xi)$ and $\Delta_2(\xi)$ with vertex at the point ξ. We denote by $R(f, \xi, S)$ the set of all recurring values of a function $f(z)$ on the set S, i.e., the set of all real numbers $a \in R$ such that $a = f(z_n^a)$, $n \in N$, for some sequence $\{z_n^a\}$ from S for which $\lim\limits_{n\to\infty} z_n^a = \xi$. A point $\xi \in \Gamma$ is a Plesner point if for every Stolz angle $\Delta(\xi)$ the equality $C(f, \xi, \Delta(\xi)) = \overline{R}$ holds. The set of all Plesner points is denoted by $I(f)$. According to the definition from [3], we denote by $B(f)$ the set of of all points $\xi \in K(f)$ such that for every angle $\Delta(\xi)$ the following is valid: $C(f, \xi, \Delta(\xi)) \neq \overline{R}$ and $C(f, \xi, \Delta(\xi))$ contains more than one element.

Adhering the definition given in [7], we call a point $\xi \in \Gamma$ a Lindelöf point of a real-valued function $f(z)$, defined in D, if for every angles $\Delta_1(\xi)$ and $\Delta_2(\xi)$ the relation $C(f, \xi, \Delta_1(\xi)) = C(f, \xi, \Delta_2(\xi)) \neq \overline{R}$ is valid. The set of Lindelöf points is denoted by $L(f)$. According to [3], a point $\xi \in \Gamma$ is called a refined Lindelöf point of a real-valued function $f(z)$, if

a) $C(f, \xi, \Delta_1(\xi)) = C(f, \xi, \Delta_2(\xi)) \neq \overline{R}$ for every angles $\Delta_1(\xi)$ and $\Delta_2(\xi)$,

b) $C(f, \xi, \Delta(\xi))$ contains more than one point for every angle $\Delta(\xi)$,

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