



Physics for Pre-medical students

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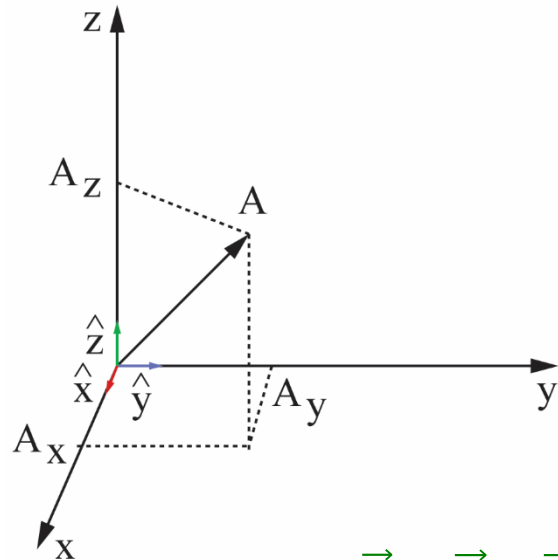
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Mathematical Methods

Coordinate Systems, Points, Vectors



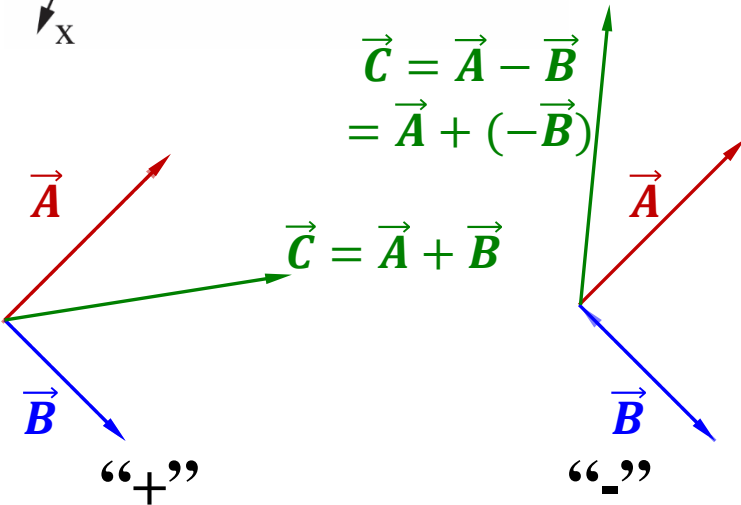
A **vector** in a coordinate system is a directed line between two points. It has **magnitude** and **direction**. Once we define a coordinate origin, each particle in a system has a **position vector** (e.g. \vec{A}) associated with its location in space drawn from the origin to the physical coordinates of the particle (e.g. (A_x, A_y, A_z)):

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

The position vectors clearly depend on the choice of coordinate origin. However, the **difference vector** or **displacement vector** between two position vectors does **not** depend on the coordinate origin. To see this, let us consider the **addition** of two vectors:

$$\vec{A} + \vec{B} = \vec{C}$$

Note that vector addition proceeds by putting the tail of one at the head of the other, and constructing the vector that completes the triangle.





Vector

If we are given a vector in terms of its **length** (magnitude) and **orientation** (direction angle(s)) then we must evaluate its cartesian components before we can add them (for example, in 2D):

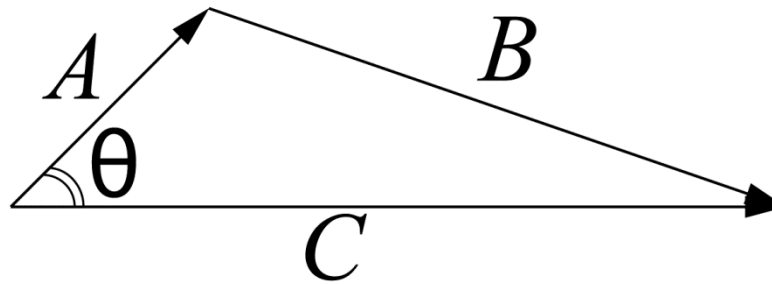
$$A_x = |\vec{A}| \cos(\theta_A)$$

$$A_y = |\vec{A}| \sin(\theta_A)$$

$$B_x = |\vec{B}| \cos(\theta_B)$$

$$B_y = |\vec{B}| \sin(\theta_B)$$

This process is called **decomposing** the vector into its cartesian components.



The **difference** between two vectors is defined by the addition law. Subtraction is just adding the negative of the vector in question, that is, the vector with the **same** magnitude but the **opposite** direction. This is consistent with the notion of adding or subtracting its components.

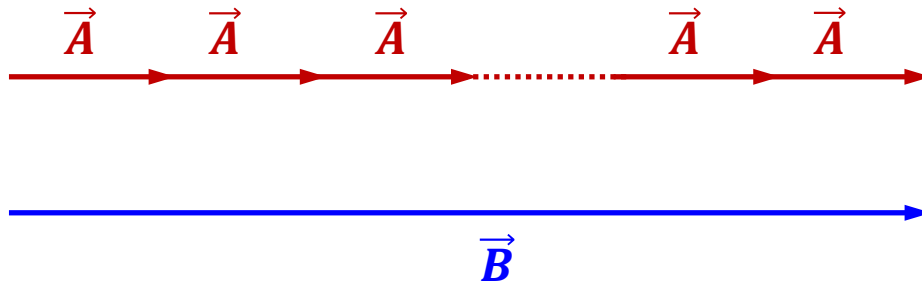


Scalar

When we reconstruct a vector from its components, we are just using the law of vector addition itself, by **scaling** some special vectors called **unit vectors** and then adding them. Unit vectors are (typically perpendicular) vectors that define the essential directions and orientations of a coordinate system and have unit length. Scaling them involves multiplying these unit vectors by a number that represents the magnitude of the vector component. This scaling number has no direction and is called a **scalar**.

$$\vec{B} = C\vec{A}$$

where C is a scalar (number) and \vec{A} is a vector. In this case, $\vec{A} \parallel \vec{B}$ (\vec{A} is parallel to \vec{B}).





Let's define products that **multiply** two vectors together

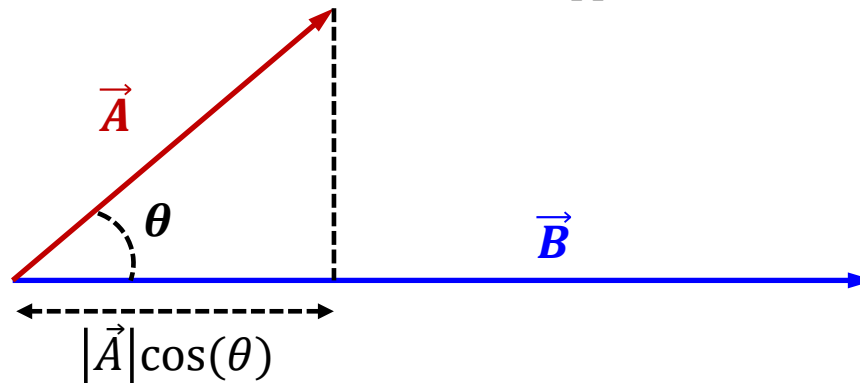
The first product creates a scalar (ordinary number with magnitude but no direction) out of two vectors and is therefore called a **scalar product** or (because of the multiplication symbol chosen) a **dot product**.

$$|\vec{A}| = +\sqrt{\vec{A} \cdot \vec{A}}$$

$$\vec{A} \cdot \vec{B} = A_x * B_x + A_y * B_y \dots = |\vec{A}| |\vec{B}| \cos(\theta_{AB})$$

A scalar product is the length of one vector (either one, say $|\vec{A}|$) times the component of the other vector ($|\vec{B}| \cos(\theta_{AB})$) that points in the same direction as the vector \vec{A} .

This product is *symmetric* and *commutative* (\vec{A} and \vec{B} can appear in either order or role).



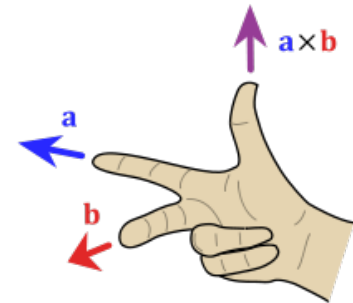
A vector product

The other product multiplies two vectors in a way that creates a third vector. It is called a **vector product** or (because of the multiplication symbol chosen) a **cross product**.

$$\vec{A} \times \vec{B} = (A_x * B_y - A_y * B_x)\hat{z} + (A_y * B_z - A_z * B_y)\hat{x} + (A_z * B_x - A_x * B_z)\hat{y}$$

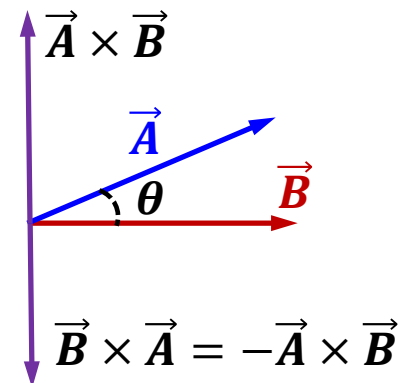
$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin(\theta_{AB})$$

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$



Let's define the direction of the cross product using the **right hand rule**:

Let the fingers of your right hand lie along the direction of the first vector in a cross product (say \vec{A} below). Let them curl naturally through the small angle (observe that there are two, one of which is larger than π and one of which is less than π) into the direction of \vec{B} . The erect thumb of your right hand then points in the general direction of the cross product vector – it at least indicates which of the two perpendicular lines should be used as a direction, unless your thumb and fingers are all double jointed or your bones are missing or you used your left-handed right hand or something.

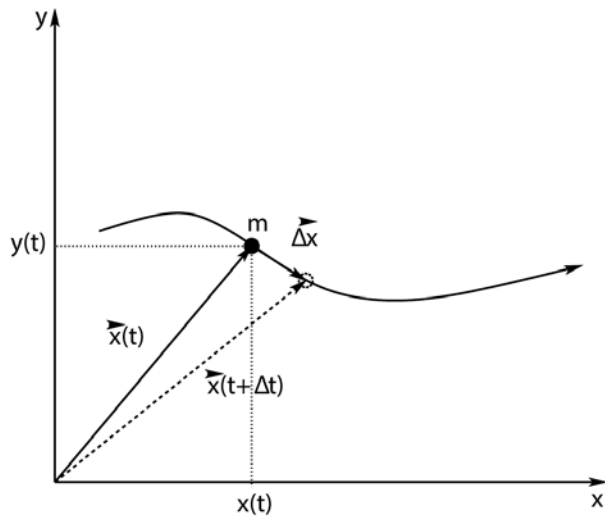




Lecture 1. Newton's Laws

Coordinates

Physics is the study of *dynamics*. Dynamics is the description of the actual forces of nature that, we believe, underlie the causal structure of the Universe and are responsible for its *evolution in time*. We are about to embark upon the intensive study of a simple description of nature that introduces the concept of a *force*, due to Isaac Newton. A force is considered to be the *causal agent* that produces the effect of *acceleration* in any massive object, altering its dynamic state of motion.



- a) meters – the SI units of length
- b) seconds – the SI units of time
- c) kilograms – the SI units of mass

Coordinatized visualization of the motion of a particle of mass m along a trajectory $\vec{x}(t)$. Note that in a short time Δt the particle's position changes from $\vec{x}(t)$ to $\vec{x}(t+\Delta t)$.

$$\vec{x}(t) = x(t)\hat{x} + y(t)\hat{y}$$



Lecture 1. Newton's Laws

Velocity

The *average* velocity of the particle is by definition the vector change in its position $\Delta\vec{x}$ in some time Δt divided by that time:

$$\vec{v}_{av} = \frac{\Delta\vec{x}}{\Delta t}$$

Sometimes average velocity is useful, but often, even usually, it is not. It can be a rather poor measure for how fast a particle is actually moving at any given time, especially if averaged over times that are long enough for interesting changes in the motion to occur.

The *instantaneous* velocity vector is the time-derivative of the position vector:

$$\vec{v}(t) = \lim_{\Delta t \rightarrow 0} \frac{\vec{x}(t + \Delta t) - \vec{x}(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta\vec{x}}{\Delta t} = \frac{d\vec{x}}{dt}$$

Speed is defined to be the *magnitude* of the velocity vector:

$$v(t) = |\vec{v}(t)|$$



Lecture 1. Newton's Laws

Acceleration

To see how the velocity changes in time, we will need to consider the acceleration of a particle, or the rate at which the velocity changes. As before, we can easily define an *average acceleration* over a possibly long time interval Δt as:

$$\vec{a}_{av} = \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt}$$

The acceleration that really matters is (again) the limit of the average over very *short* times; the time derivative of the velocity. This limit is thus defined to be the *instantaneous* acceleration:

$$\vec{a}(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2 \vec{x}}{dt^2}$$



Lecture 1. Newton's Laws

Newton's Laws

- a) **Law of Inertia:** Objects at rest or in uniform motion (at a constant velocity) in an *inertial reference frame* remain so unless acted upon by an unbalanced (net, total) force. We can write this algebraically as:

$$\vec{F} = \sum_i \vec{F}_i = 0 = m\vec{a} = m \frac{d\vec{v}}{dt} \Rightarrow \vec{v} = \text{constant vector}$$

- b) **Law of Dynamics:** The total force applied to an object is directly proportional to its acceleration in an *inertial reference frame*. The constant of proportionality is called the **mass** of the object. We write this algebraically as:

$$\vec{F} = \sum_i \vec{F}_i = m\vec{a} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt}$$

where we introduce the *momentum* of a particle, $\vec{p} = m\vec{v}$.

- c) **Law of Reaction:** If object A exerts a force \vec{F}_{AB} on object B *along a line connecting the two objects*, then object B exerts an equal and opposite reaction force of $\vec{F}_{AB} = -\vec{F}_{BA}$ on object A. We write this algebraically as:

$$\vec{F}_{ij} = -\vec{F}_{ji} \text{ (or) } \sum_{i,j} \vec{F}_{ij} = 0$$

where i and j are arbitrary particle labels. The latter form will be useful to us later; it means that the sum of all *internal* forces between particles in any closed system of particles cancels!



Lecture 1. Newton's Laws

Forces

Classical dynamics at this level, in a nutshell, is very simple. Find the total force on an object. Use Newton's second law to obtain its acceleration (as a differential equation of motion). Solve the equation of motion by direct integration or otherwise for the position and velocity.

The next most important problem is: how do we evaluate the total force?

There are *fundamental* forces – *elementary* forces that we call “laws of nature” because the forces themselves aren't caused by some other force, they are themselves the actual causes of dynamical action in the visible Universe.

The Forces of Nature (strongest to weakest):

- a) Strong Nuclear (bound together the quarks, protons and neutrons)
- b) Electromagnetic (combines the positive nucleus with electrons)
- c) Weak Nuclear (acts at very short range. This force can cause e.g. neutrons to give off an electron and turn into a proton)
- d) Gravity



Lecture 1. Newton's Laws

Force Rules

- a) **Gravity** (near the surface of the earth):

$$F_g = mg,$$

$$g \approx 9,81 \frac{\text{meter}}{\text{second}^2} \approx 10 \frac{\text{meter}}{\text{second}^2}$$

- b) **The Spring** (Hooke's Law) in one dimension:

$$F_x = -k\Delta x$$

- c) **The Normal Force:**

$$F_{\perp} = N$$

- d) **Tension** in an Acme (massless, unstretchable, unbreakable) string:

$$F_S = T$$

- e) **Static Friction:**

$$f_S \leq \mu_s N$$

- f) **Kinetic Friction:**

$$f_k = \mu_k N$$

- g) **Fluid Forces, Pressure:** A fluid in contact with a solid surface (or anything else) in general exerts a force on that surface that is related to the pressure of the fluid:

$$F_p = PA$$

- h) **Drag Forces:**

$$F_d = -bv^n$$



Lecture 1. Newton's Laws

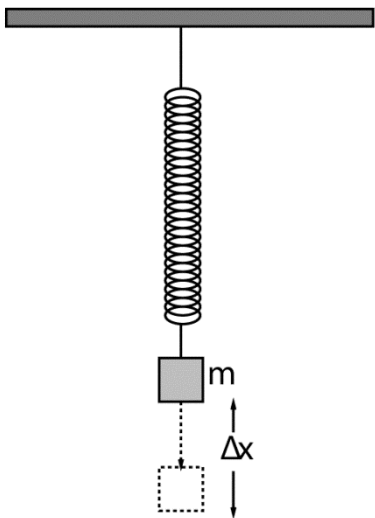
Force Balance – Static Equilibrium

If all of the forces acting on an object balance:

$$\vec{F}_{tot} = \sum_i \vec{F}_i = m\vec{a} = 0$$

Example: Spring and Mass in *Static Force Equilibrium*

Suppose we have a mass m hanging on a spring with spring constant k such that the spring is stretched out some distance Δx from its unstretched length.



A mass m hangs on a spring with spring constant k . We would like to compute the amount Δx by which the string is stretched when the mass is at rest in static force equilibrium.

$$\sum F_x = -k(x - x_0) - mg = ma_x$$

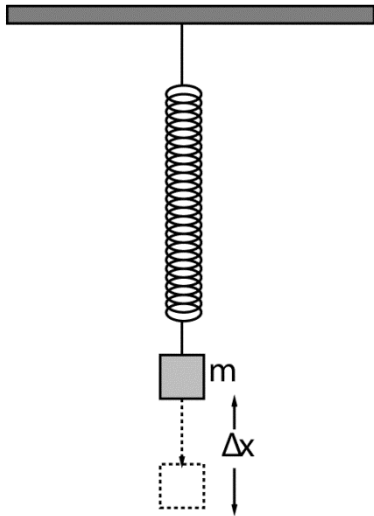
or (with $\Delta x = x - x_0$, so that Δx is negative as shown)

$$a_x = -\frac{k}{m}\Delta x - g$$



Lecture 1. Newton's Laws

Force Balance – Static Equilibrium



In static equilibrium, $a_x = 0$ (and hence, $F_x = 0$) and we can solve for Δx :

$$a_x = -\frac{k}{m}\Delta x - g = 0$$

$$\frac{k}{m}\Delta x = g$$

$$\Delta x = \frac{mg}{k}$$



Lecture 1. Newton's Laws

Simple Motion in One Dimension

A mass m at rest is dropped from a height H above the ground at time $t = 0$; what happens to the mass as a function of time?

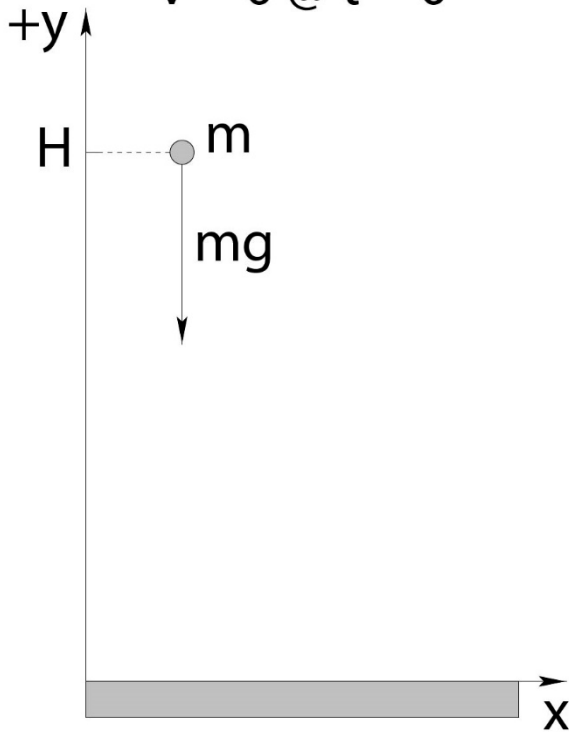
1. You must select a coordinate system to use to describe what happens.
2. You must write Newton's Second Law in the coordinate system for all masses, being sure to include all forces or force rules that contribute to its motion.
3. You must solve Newton's Second Law to find the accelerations of all the masses (equations called the equations of motion of the system).
4. You must solve the equations of motion to find the trajectories of the masses, their positions as a function of time, as well as their velocities as a function of time if desired.
5. Finally, armed with these trajectories, you must answer all the questions the problem poses using algebra and reason



Lecture 1. Newton's Laws

Example: A Mass Falling from Height H

$v = 0$ @ $t = 0$



Draw in all of the forces that act on the mass as proportionate vector arrows in the direction of the force.

$$\vec{F} = -mg\hat{y}$$

or if you prefer, you can write the dimension-labelled scalar equation for the magnitude of the force in the y-direction:

$$F_y = -mg$$

$$F_y = -mg = ma_y$$

$$ma_y = -mg$$

$$a_y = -g$$

$$\frac{d^2y}{dt^2} = \frac{dv_y}{dt} = -g$$

where $g = 10 \text{ m/second}^2$



Lecture 1. Newton's Laws

Example: A Mass Falling from Height H

The last line (the algebraic expression for the acceleration) is called the equation of motion for the system

$$\frac{dv_y}{dt} = -g \text{ Next, multiply both sides by } dt \text{ to get:}$$

$$dv_y = -g dt \text{ Then integrate both sides:}$$

$$\int dv_y = - \int g dt \text{ doing the indefinite integrals to get:}$$

$$v_y(t) = -g \cdot t + C$$

The final C is the constant of integration of the indefinite integrals. We have to evaluate it using the given (usually initial) conditions. In this case we know that:

$$v_y(0) = -g \cdot 0 + C = C = 0$$

Thus:

$$v_y(t) = -gt$$

We now know the velocity of the dropped ball as a function of time!



Lecture 1. Newton's Laws

Example: A Mass Falling from Height H

However, the solution to the dynamical problem is the trajectory function, $y(t)$. To find it, we repeat the same process, but now use the definition for v_y in terms of y :

$$\frac{dy}{dt} = v_y(t) = -gt \text{ Multiply both sides by } dt \text{ to get:}$$

$$dy = -gt \, dt \text{ Next, integrate both sides:}$$

$$\int dy = - \int gt \, dt \text{ to get:}$$

$$y(t) = -\frac{1}{2}gt^2 + D$$

The final D is again the constant of integration of the indefinite integrals. We again have to evaluate it using the given (initial) conditions in the problem. In this case we know that:

$$y(0) = -\frac{1}{2}g0^2 + D = D = H$$

because we dropped it from an initial height $y(0) = H$. Thus:

$$y(t) = -\frac{1}{2}gt^2 + H$$

and we know everything there is to know about the motion!



Lecture 1. Newton's Laws

Example: A Mass Falling from Height H

Finally, we have to answer any questions that the problem might ask! Here are a couple of common questions you can now answer using the solutions you just obtained:

- a) How long will it take for the ball to reach the ground?
- b) How fast is it going when it reaches the ground?

To answer the first one, we use a bit of algebra. “The ground” is (recall) $y = 0$ and it will reach there at some specific time (the time we want to solve for) t_g .

We write the condition that it is at the ground at time t_g :

$$y(t_g) = -\frac{1}{2}gt^2 + H = 0$$

If we rearrange this and solve for t_g we get:

$$t_g = \pm \sqrt{\frac{2H}{g}}$$



Lecture 1. Newton's Laws

Example: A Mass Falling from Height H

To find the speed at which it hits the ground, one can just take our correct (future) time and plug it into v_y ! That is:

$$v_g = v_y(t_g) = -gt_g = -g \sqrt{\frac{2H}{g}} = -\sqrt{2gH}$$

Note well that it is going down (in the negative y direction) when it hits the ground.

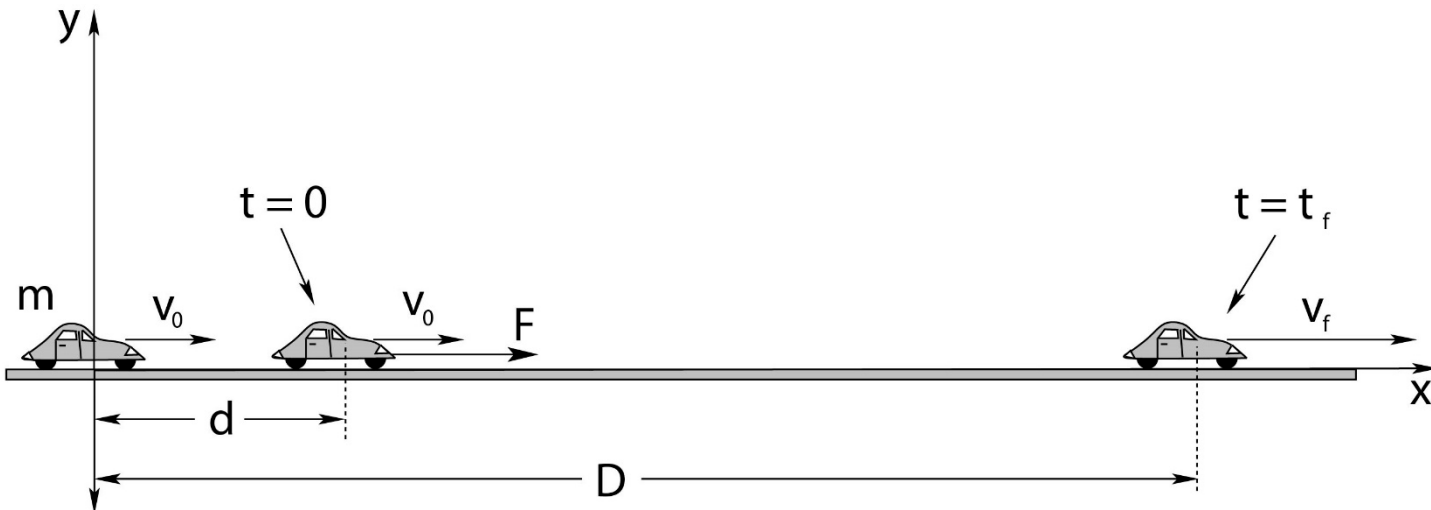


Lecture 1. Newton's Laws

Example: A Constant Force in One Dimension

A car of mass m is travelling at a constant speed v_0 as it enters a long, nearly straight merge lane. A distance d from the entrance, the driver presses the accelerator and the engine exerts a constant force of magnitude F on the car.

- a) How long does it take the car to reach a final velocity $v_f > v_0$?
- b) How far (from the entrance) does it travel in that time?





Lecture 1. Newton's Laws

Example: A Constant Force in One Dimension

We will write Newton's Second Law and solve for the acceleration (obtaining an equation of motion). Then we will integrate twice to find first $v_x(t)$ and then $x(t)$.

$$F = ma_x$$

$$a_x = \frac{F}{m} = a_0 \quad (\text{a constant})$$

$$\frac{dv_x}{dt} = a_0$$

Next, multiply through by dt and integrate both sides:

$$v_x(t) = \int dv_x = \int a_0 dt = a_0 t + V = \frac{F}{m} t + V$$

V is a constant of integration that we will evaluate below.

Note that if $a_0 = F/m$ was not a constant (say that $F(t)$ is a function of time) then we would have to do the integral:

$$v_x(t) = \int \frac{F(t)}{m} dt = \frac{1}{m} \int F(t) dt = ???$$



Lecture 1. Newton's Laws

Example: A Constant Force in One Dimension

At time $t = 0$, the velocity of the car in the x -direction is v_0 , so $V = v_0$ and:

$$v_x(t) = a_0 t + v_0 = \frac{dx}{dt}$$

We multiply this equation by dt on both sides, integrate, and get:

$$x(t) = \int dx = \int (a_0 t + v_0) dt = \frac{1}{2} a_0 t^2 + v_0 t + x_0$$

where x_0 is the constant of integration. We note that at time $t = 0$, $x(0) = d$, so $x_0 = d$. Thus:

$$x(t) = \frac{1}{2} a_0 t^2 + v_0 t + d$$

$$v_x(t) = a_0 t + v_0$$

$$x(t) = \frac{1}{2} a_0 t^2 + v_0 t + x_0$$



Lecture 1. Newton's Laws

Motion in Two Dimensions

The idea of motion in two or more dimensions is very simple. Force is a vector, and so is acceleration. Newton's Second Law is a recipe for taking the total force and converting it into a differential equation of motion:

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{\vec{F}_{tot}}{m}$$

If we write the equation of motion out in components:

$$a_x = \frac{d^2x}{dt^2} = \frac{F_{tot,x}}{m}$$

$$a_y = \frac{d^2y}{dt^2} = \frac{F_{tot,y}}{m}$$

$$a_z = \frac{d^2z}{dt^2} = \frac{F_{tot,z}}{m}$$

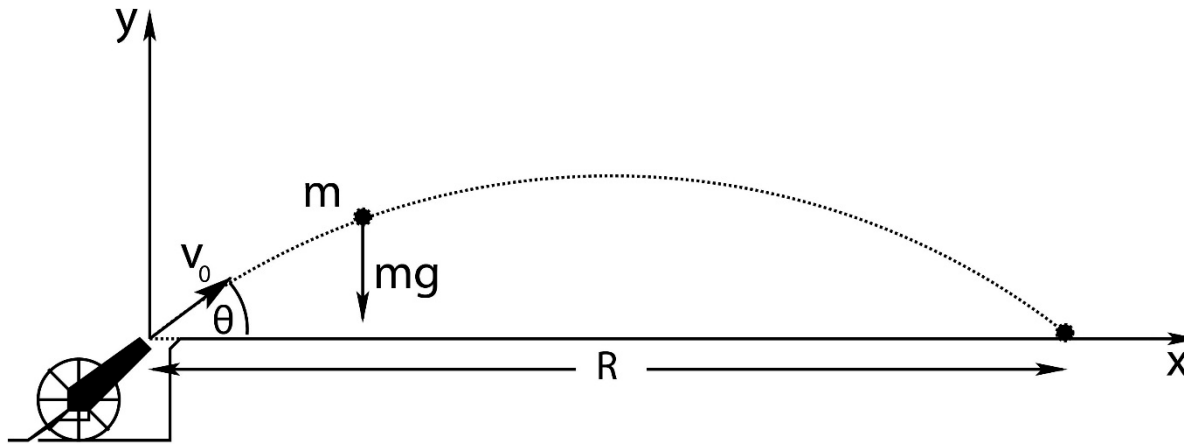
we will often reduce the complexity of the problem from a “three dimensional problem” to three “one dimensional problems”.

Select a coordinate system in which one of the coordinate axes is aligned with the total force.



Lecture 1. Newton's Laws

Example: Trajectory of a Cannonball



An idealized cannon, neglecting the drag force of the air. Let x be the horizontal direction and y be the vertical direction, as shown. Note well that $\vec{F}_g = -mg\vec{y}$ points along one of the coordinate directions while $F_x = (F_z =) 0$ in this coordinate frame.

A cannon fires a cannonball of mass m at an initial speed v_0 at an angle θ with respect to the ground as shown in figure. Find:

- The time the cannonball is in the air.
- The range of the cannonball.



Lecture 1. Newton's Laws

Example: Trajectory of a Cannonball

Newton's Second Law for both coordinate directions:

$$F_x = ma_x = 0$$

$$F_y = ma_y = m \frac{d^2y}{dt^2} = -mg$$

We divide each of these equations by m to obtain two equations of motion, one for x and the other for y :

$$a_x = 0$$

$$a_y = -g$$

We solve them independently. In x :

$$a_x = \frac{dv_x}{dt} = 0$$

The derivative of any constant is zero, so the x -component of the velocity does not change in time. We find the initial (and hence constant) component using trigonometry:

$$v_x(t) = v_{0x} = v_0 \cos \theta$$



Lecture 1. Newton's Laws

Example: Trajectory of a Cannonball

We then write this in terms of derivatives and solve it:

$$v_x(t) = \frac{dx}{dt} = v_0 \cos(\theta)$$

$$dx = v_0 \cos(\theta) dt$$

$$\int dx = v_0 \cos(\theta) \int dt$$

$$x(t) = v_0 \cos(\theta) t + C$$

We evaluate C (the constant of integration) from our knowledge that in the coordinate system we selected, $x(0) = 0$ so that $C = 0$. Thus:

$$x(t) = v_0 \cos(\theta) t$$



Lecture 1. Newton's Laws

Example: Trajectory of a Cannonball

The solution in y is more or less identical to the solution that we obtained above dropping a ball, except the constants of integration are different:

$$a_y = \frac{dv_y}{dt} = -g$$

$$dv_y = -g dt$$

$$\int dv_y = - \int g dt$$

$$v_y(t) = -gt + C'$$

For this problem, we know from trigonometry that:

$$v_y(0) = v_0 \sin(\theta)$$

so that $C' = v_0 \sin(\theta)$ and:

$$v_y(t) = -gt + v_0 \sin(\theta)$$



Lecture 1. Newton's Laws

Example: Trajectory of a Cannonball

We write v_y in terms of the time derivative of y and integrate:

$$\frac{dy}{dt} = v_y(t) = -gt + v_0 \sin(\theta)$$

$$dy = (-gt + v_0 \sin(\theta)) dt$$

$$\int dy = \int (-gt + v_0 \sin(\theta)) dt$$

$$y(t) = -\frac{1}{2}gt^2 + v_0 \sin(\theta)t + D$$

Again we use $y(0) = 0$ in the coordinate system we selected to set $D = 0$ and get:

$$y(t) = -\frac{1}{2}gt^2 + v_0 \sin(\theta)t$$



Lecture 1. Newton's Laws

Example: Trajectory of a Cannonball

Collecting the results from above, our overall solution is thus:

$$x(t) = v_0 \cos(\theta) t$$

$$y(t) = -\frac{1}{2}gt^2 + v_0 \sin(\theta)t$$

$$v_x(t) = v_{0x} = v_0 \cos \theta$$

$$v_y(t) = -gt + v_0 \sin(\theta)$$

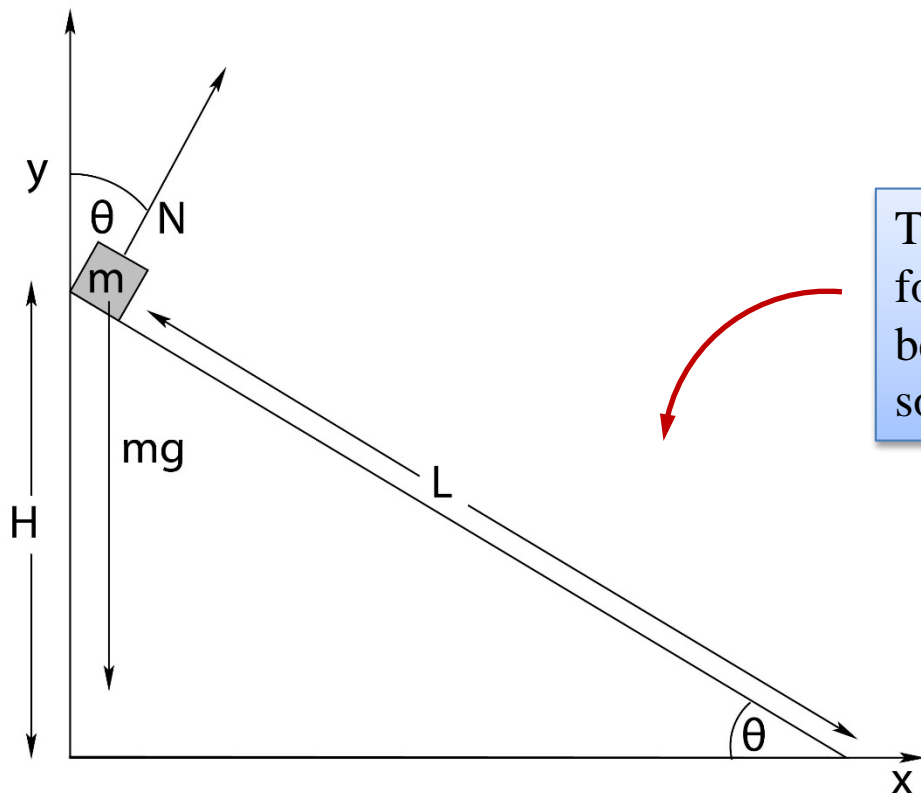
We know exactly where the cannonball is at all times, and we know exactly what its velocity is as well.



Lecture 1. Newton's Laws

The Inclined Plane

In this problem we will talk about a new force, the *normal* force. Recall from above that the normal force is whatever magnitude it needs to be to prevent an object from moving in to a solid surface, and is always perpendicular (normal) to that surface in direction.



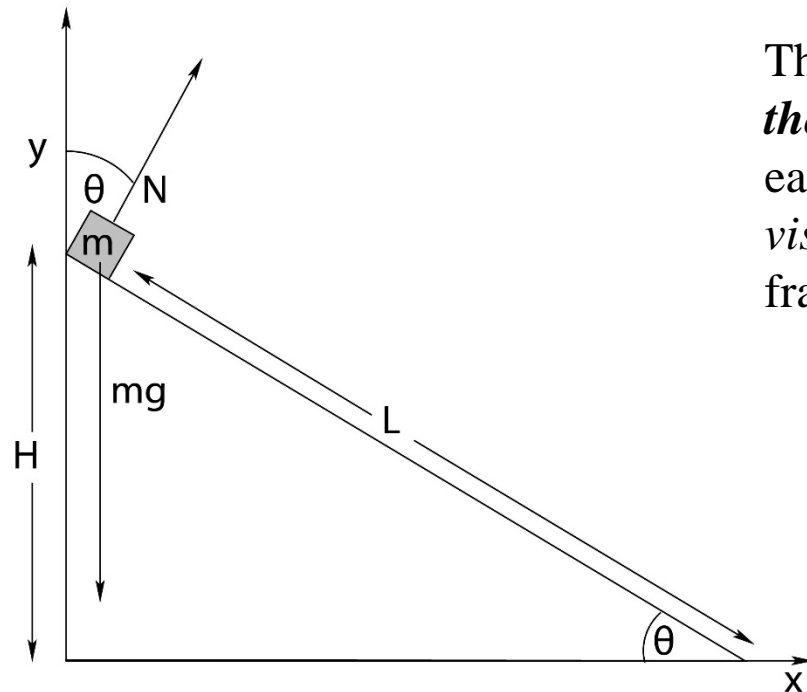
This is the naive/wrong coordinate system to use for the inclined plane problem. The problem can be solved in this coordinate frame, but the solution (as you can see) would be quite difficult.



Lecture 1. Newton's Laws

The Inclined Plane

A block m rests on a plane inclined at an angle of θ with respect to the horizontal. There is no friction, but the plane exerts a normal force on the block that keeps it from falling straight down. At time $t = 0$ it is released (at a height $H = L\sin(\theta)$ above the ground), and we might then be asked any of the “usual” questions – how long does it take to reach the ground, how fast is it going when it gets there and so on.



The motion we expect is for the block to *slide down the incline*, and for us to be able to solve the problem easily we have to use our *intuition* and ability to *visualize* this motion to select the best coordinate frame.



Lecture 1. Newton's Laws

The Inclined Plane

Let's try to decompose these forces in terms of our coordinate system:

$$N_x = N \sin \theta$$

$$N_y = N \cos \theta$$

where $N = |\vec{N}|$ is the (unknown) magnitude of the normal force.

We then add up the total forces in each direction and write Newton's Second Law for each direction's total force :

$$F_x = N \sin \theta = ma_x$$

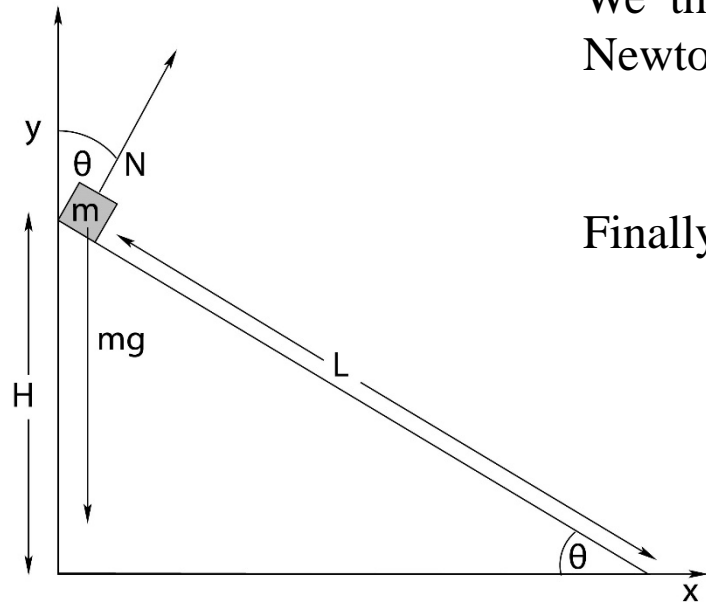
$$F_y = N \cos \theta - mg = ma_y$$

Finally, we write our equations of motion for each direction:

$$a_x = \frac{N \sin \theta}{m}$$

$$a_y = \frac{N \cos \theta - mg}{m}$$

Unfortunately, we cannot solve these two equations as written yet. That is because we do not know the value of N ; it is in fact something we need to solve for!





Lecture 1. Newton's Laws

The Inclined Plane

To solve them we need to add a condition on the solution, expressed as an equation. The condition we need to add is that the motion is down the incline, that is, at all times:

$$\frac{y(t)}{L \cos \theta - x(t)} = \tan \theta$$

That means that:

$$\begin{aligned} y(t) &= (L \cos \theta - x(t)) \tan \theta \\ \frac{dy(t)}{dt} &= -\frac{dx(t)}{dt} \tan \theta \\ \frac{d^2 y(t)}{dt^2} &= -\frac{d^2 x(t)}{dt^2} \tan \theta \\ a_y &= -a_x \tan \theta \end{aligned}$$

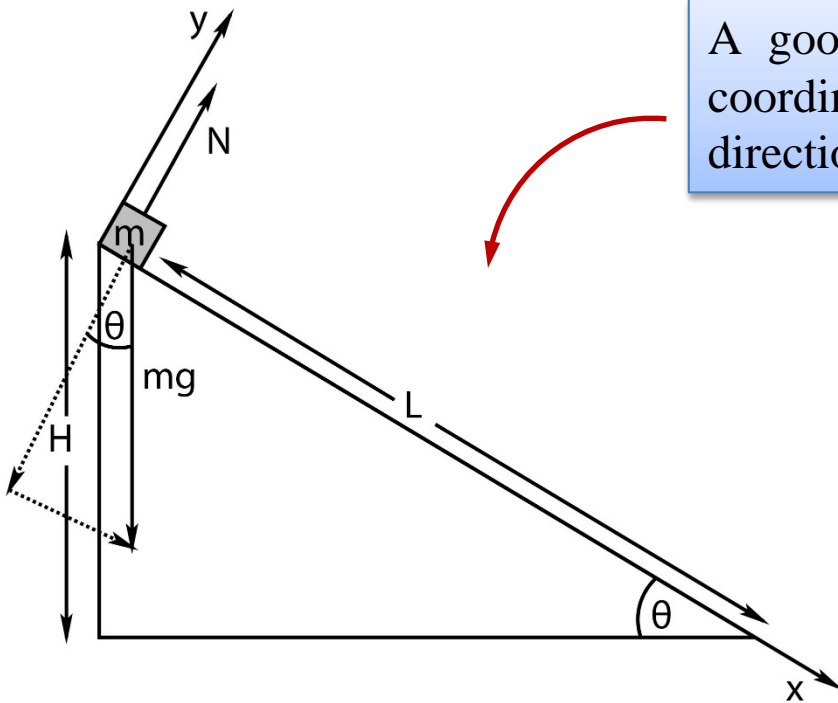
We can use this relation to eliminate (say) a_y from the equations above, solve for a_x , then backsubstitute to find a_y .

The solutions we get will be so very complicated (at least compared to choosing a better frame), with both x and y varying nontrivially with time!



Lecture 1. Newton's Laws

The Inclined Plane



A good choice of coordinate frame has (say) the x -coordinate lined up with the total force and hence direction of motion.

We can decompose the forces in this coordinate system, but now we need to find the components of the gravitational force as $\vec{N} = N\hat{y}$ is easy! Furthermore, we know that $a_y = 0$ and hence $F_y = 0$.

$$F_x = mg \sin \theta = ma_x$$

$$F_y = N - mg \cos \theta = ma_y = 0$$

We can immediately solve the y equation for:

$$N = mg \cos \theta$$

and write the equation of motion for the x -direction: $a_x = g \sin \theta$ which is a constant.

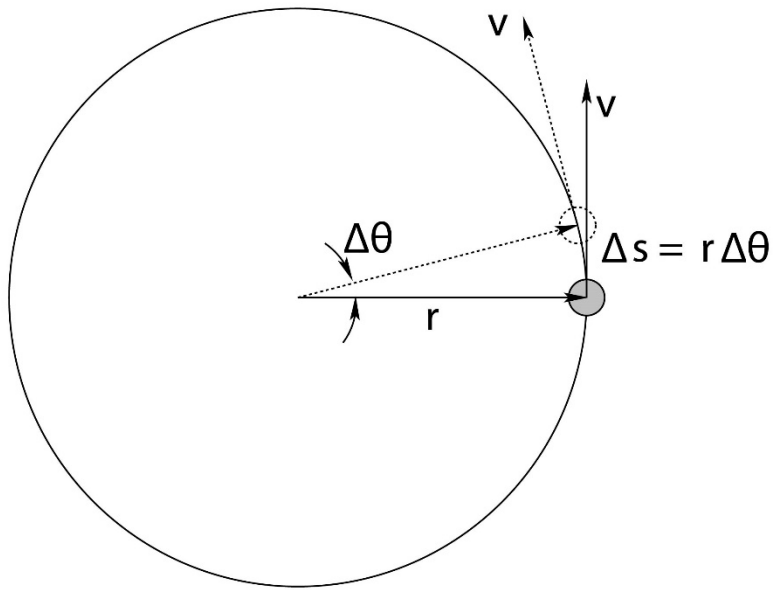
From this point on the solution should be familiar – since $v_y(0) = 0$ and $y(0) = 0$, $y(t) = 0$ and we can **ignore** y altogether and the problem is now **one dimensional!**

See if you can find how long it takes for the block to reach bottom, and how fast it is going when it gets there. You should find that $v_{bottom} = \sqrt{2gH}$



Lecture 1. Newton's Laws

Circular Motion



A small ball, moving in a circle of radius r . We are looking down from above the circle of motion at a particle moving counterclockwise around the circle. At the moment, at least, the particle is moving at a constant speed v (so that its *velocity* is always *tangent* to the circle).

The length of a circular arc is the radius times the angle subtended by the arc we can see that:

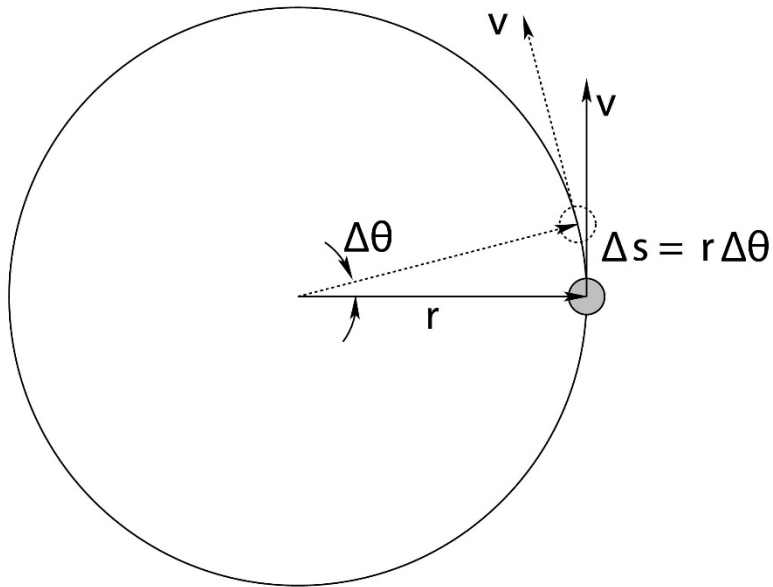
$$\Delta s = r\Delta\theta$$

Note Well! In this and all similar equations θ must be measured in **radians**, never degrees



Lecture 1. Newton's Laws

Circular Motion



The average speed v of the particle is thus this distance divided by the time it took to move it:

$$v_{avg} = \frac{\Delta s}{\Delta t} = r \frac{\Delta \theta}{\Delta t}$$

Of course, we really don't want to use average speed (at least for very long) because the speed might be varying, so we take the limit that $\Delta t \rightarrow 0$ and turn everything into derivatives, but it is much easier to draw the pictures and visualize what is going on for a small, finite Δt :

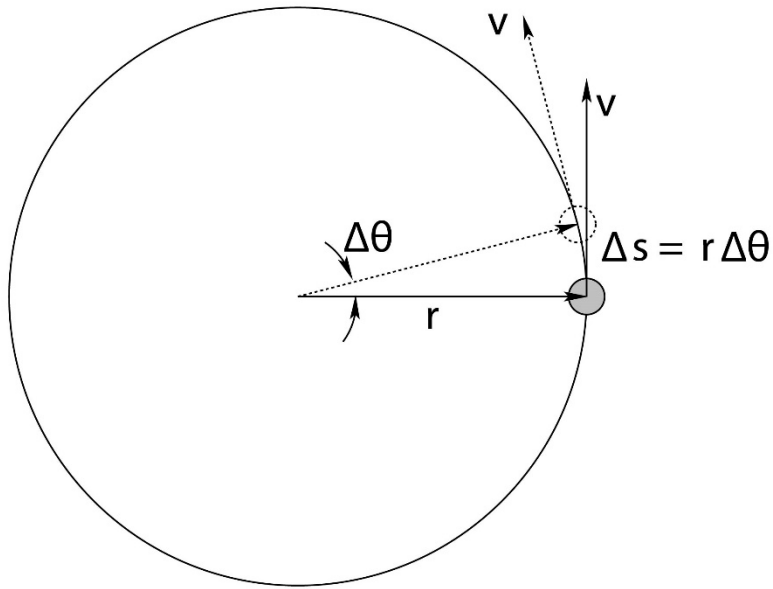
$$v = \lim_{\Delta t \rightarrow 0} r \frac{\Delta \theta}{\Delta t} = r \frac{d\theta}{dt}$$

This speed is directed tangent to the circle of motion (as one can see in the figure) and we will often refer to it as the tangential velocity.



Lecture 1. Newton's Laws

Circular Motion



$$v_t = r \frac{d\theta}{dt}$$

In this equation, we see that the speed of the particle at any instant is the radius times the rate that the angle is being swept out by the particle per unit time. This latter quantity is a very useful one for describing circular motion, or rotating systems in general.

We define it to be the **angular velocity**:

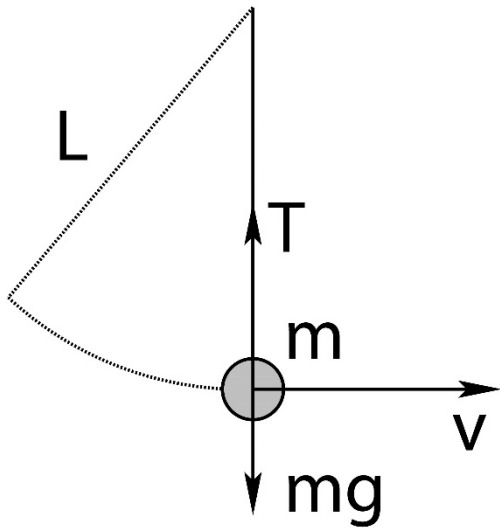
$$\omega = \frac{d\theta}{dt}$$

Thus: $v = r\omega$ or $\omega = \frac{v}{r}$



Lecture 1. Newton's Laws

Centripetal Acceleration



A ball of mass m swings down in a circular arc of radius L suspended by a string, arriving at the bottom with speed v . What is the tension in the string?

At the bottom of the trajectory, the tension T in the string points straight up and the force mg points straight down. No other forces act, so we should choose coordinates such that one axis lines up with these two forces. Let's use $+y$ vertically up, aligned with the string. Then:

$$F_y = T - mg = ma_y = m \frac{v^2}{L}$$

$$\text{or } T = mg + m \frac{v^2}{L}$$

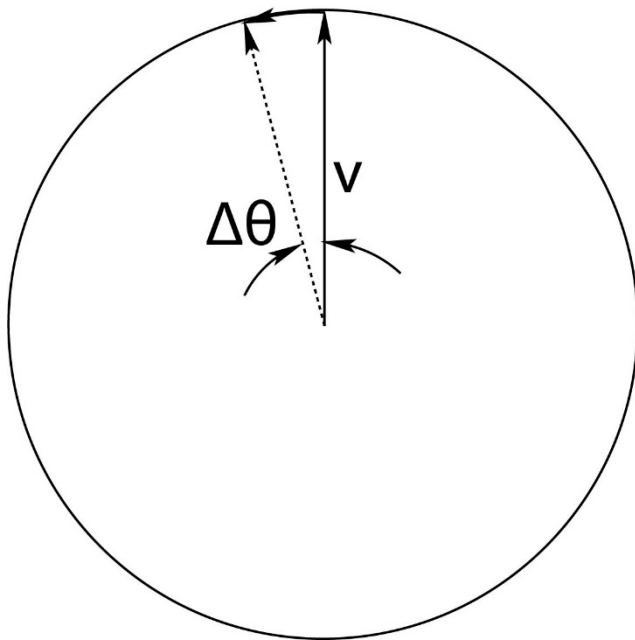
The *net* force towards the center of the circle must be algebraically equal to mv^2/r



Lecture 1. Newton's Laws

Example: Ball on a String

$$\Delta v = v \Delta \theta$$



The velocity of the particle at t and $t + \Delta t$. Note that over a very short time Δt the speed of the particle is at least approximately constant, but its *direction* varies because it always has to be perpendicular to \vec{r} , the vector from the center of the circle to the particle. The velocity swings through the *same angle* $\Delta \theta$ that the particle itself swings through in this (short) time.

In time Δt , then, the magnitude of the *change* in the velocity is:

$$\Delta v = v \Delta \theta$$

Consequently, the average magnitude of the acceleration is:

$$a_{avg} = \frac{\Delta v}{\Delta t} = v \frac{\Delta \theta}{\Delta t}$$

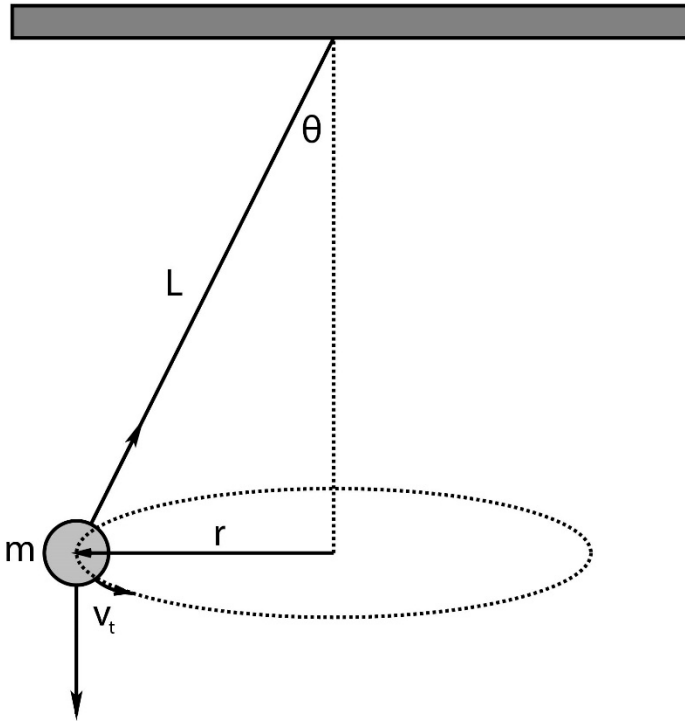
The instantaneous magnitude of the acceleration is: $a = \lim_{\Delta t \rightarrow 0} v \frac{\Delta \theta}{\Delta t} = v \frac{d\theta}{dt} = v\omega = \frac{v^2}{r} = r\omega^2$

If a particle is moving in a circle at instantaneous speed v , **then** its acceleration towards the center of that circle is v^2/r (or $r\omega^2$ if that is easier to use in a given problem).



Lecture 1. Newton's Laws

Example: Tether Ball/Conic Pendulum



Ball on a rope (a tether ball or conical pendulum). The ball sweeps out a right circular cone at an angle θ with the vertical when launched appropriately.

Suppose you hit a tether ball so that it moves in a plane circle at an angle θ at the end of a string of length L . Find T (the tension in the string) and v , the speed of the ball such that this is true.

Note well in this figure that the only “real” forces acting on the ball are gravity and the tension T in the string. Thus in the y -direction we have:

$$\sum F_y = T \cos \theta - mg = 0$$

and in the x -direction (the minus r -direction, as drawn) we have: $\sum F_x = T \sin \theta = ma_r = \frac{mv^2}{r}$

$$\text{Thus } T = \frac{mg}{\cos \theta}$$

$$v^2 = \frac{Tr \sin \theta}{m} \quad \text{or} \quad v = \sqrt{gL \sin \theta \tan \theta}$$



Lecture 1. Newton's Laws

Example: Tangential Acceleration

Sometimes we will want to solve problems where a particle speeds up or slows down while moving in a circle. Obviously, this means that there is a nonzero *tangential acceleration* changing the *magnitude* of the tangential velocity.

Let's write \vec{F} (total) acting on a particle moving in a circle in a coordinate system that rotates along with the particle – *plane polar coordinates*. The tangential direction is the $\hat{\theta}$ direction, so we will get:

$$\vec{F} = F_r \hat{r} + F_\theta \hat{\theta}$$

From this we will get two equations of motion (connecting this, at long last, to the dynamics of two dimensional motion):

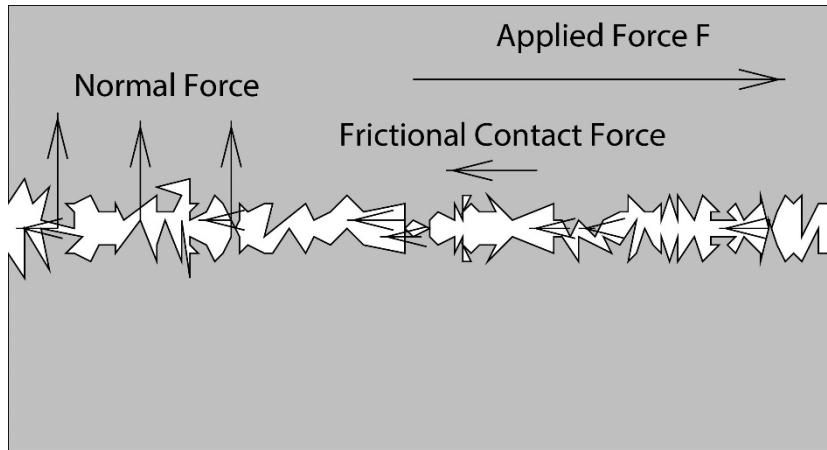
$$F_r = -m \frac{v^2}{r}$$

$$F_t = ma_t = m \frac{dv}{dt}$$



Lecture 2. Newton's Laws

Friction



Static Friction is the force exerted by one surface on another that acts *parallel* to the surfaces to *prevent the two surfaces from sliding*.

Static friction is as large as it needs to be to prevent any sliding motion, up to a maximum value, at which point the surfaces begin to slide.

The maximum force static friction can exert is proportional to both the pressure between the surfaces and the area in contact. This makes it proportional to the product of the pressure and the area, which equals the normal force. We write this as:

$$f_s \leq f_s^{\max} = \mu_s N$$

where μ_s is the *coefficient of static friction*, a dimensionless constant characteristic of the two surfaces in contact, and N is the normal force.

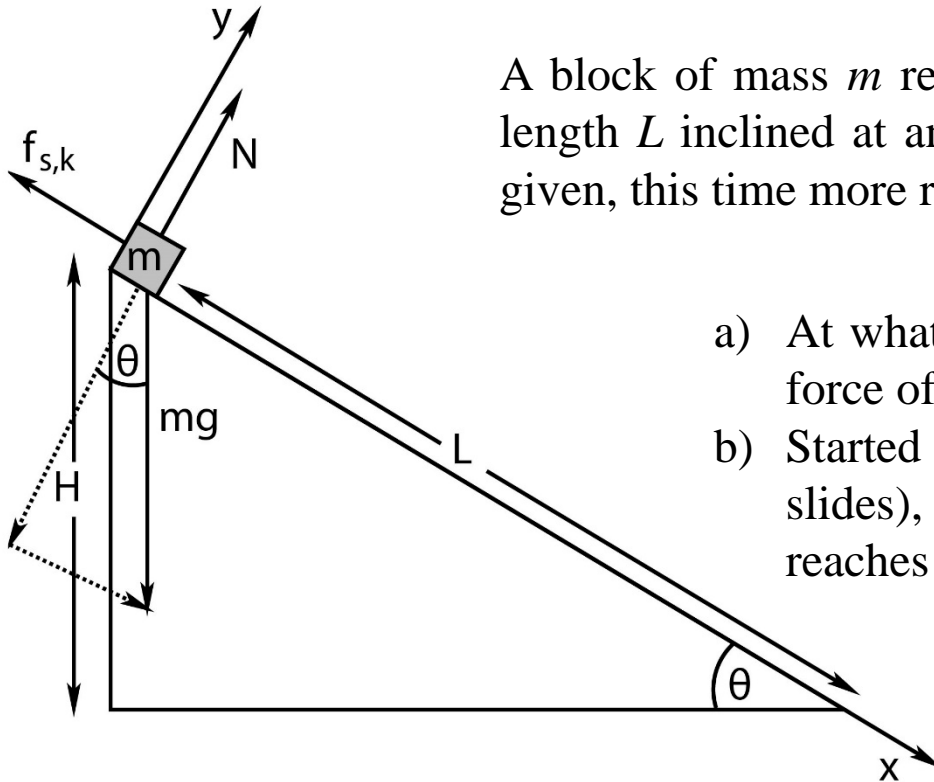
The frictional force will depend only on the total force, not the area or pressure separately:

$$f_k = \mu_k P * A = \mu_k \frac{N}{A} * A = \mu_k N$$



Lecture 2. Newton's Laws

Inclined Plane of Length L with Friction



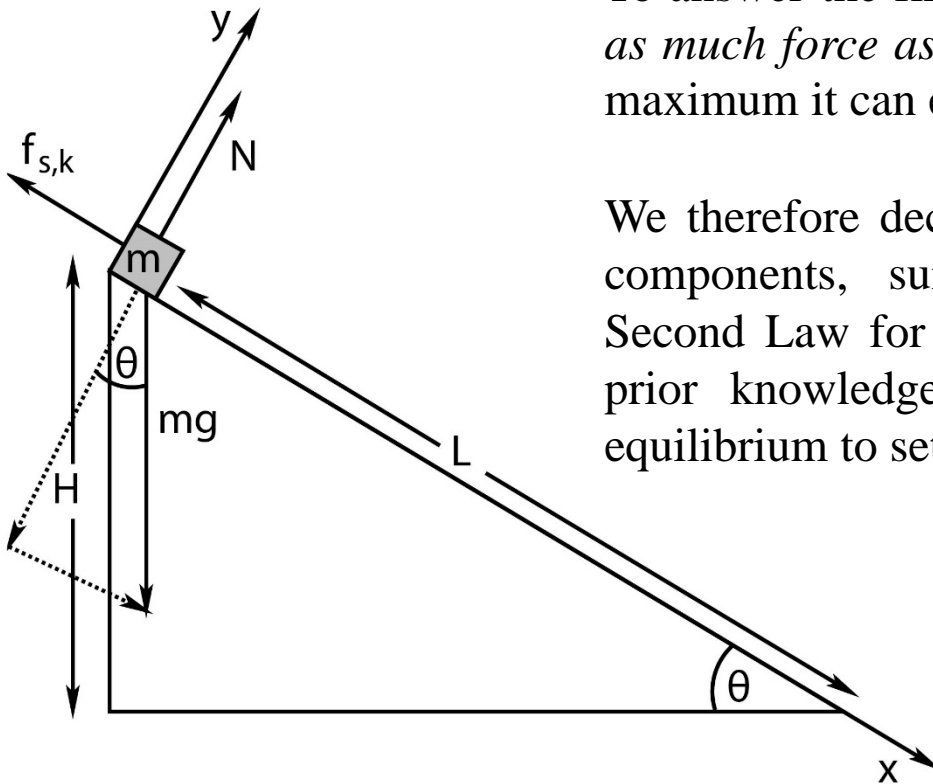
A block of mass m released from rest at time $t = 0$ on a plane of length L inclined at an angle θ relative to horizontal is once again given, this time more realistically, including the effects of *friction*.

- At what angle θ_c does the block *barely* overcome the force of static friction and slide down the incline?
- Started at rest from an angle $\theta > \theta_c$ (so it definitely slides), how fast will the block be going when it reaches the bottom?



Lecture 2. Newton's Laws

Inclined Plane of Length L with Friction



To answer the first question, we note that static friction exerts *as much force as necessary* to keep the block at rest up to the maximum it can exert, $f_s^{max} = \mu_s N$.

We therefore decompose the known force rules into x and y components, sum them componentwise, write Newton's Second Law for both vector components and finally use our prior knowledge that the system remains in static force equilibrium to set $a_x = a_y = 0$. We get:

$$\sum F_x = mg \sin \theta - f_s = 0$$

(for $\theta \leq \theta_c$ and $v(0) = 0$) and

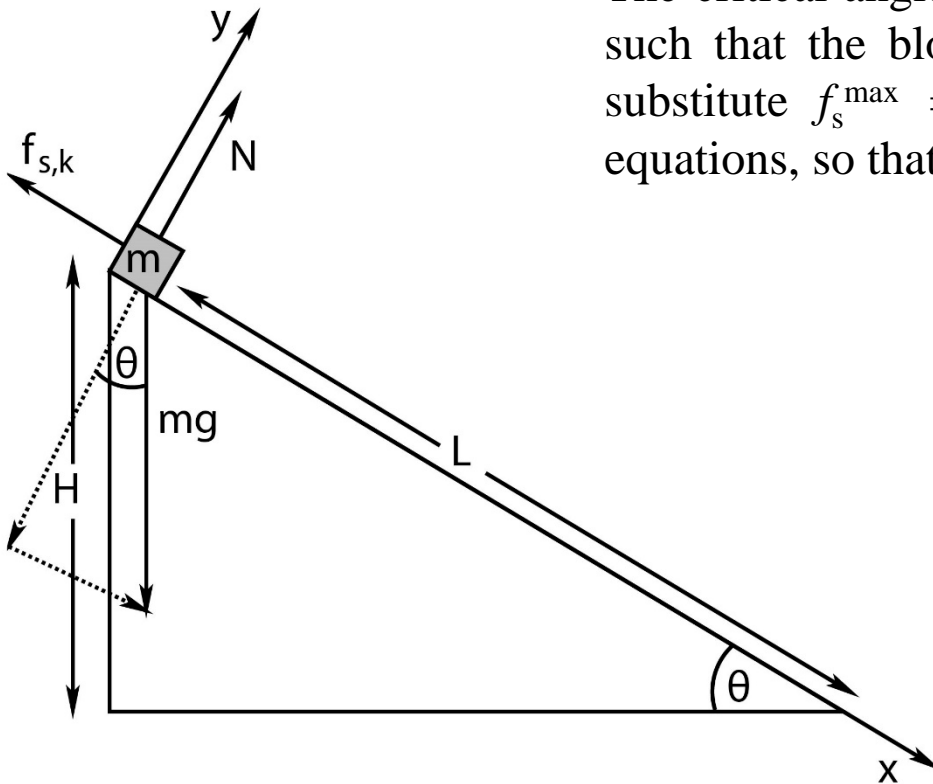
$$\sum F_y = N - mg \cos \theta = 0$$

So far, f_s is precisely what it needs to be to prevent motion: $f_s = mg \sin \theta$ while $N = mg \cos \theta$. It is true at any angle, moving or not moving, from the F_y equation



Lecture 2. Newton's Laws

Inclined Plane of Length L with Friction

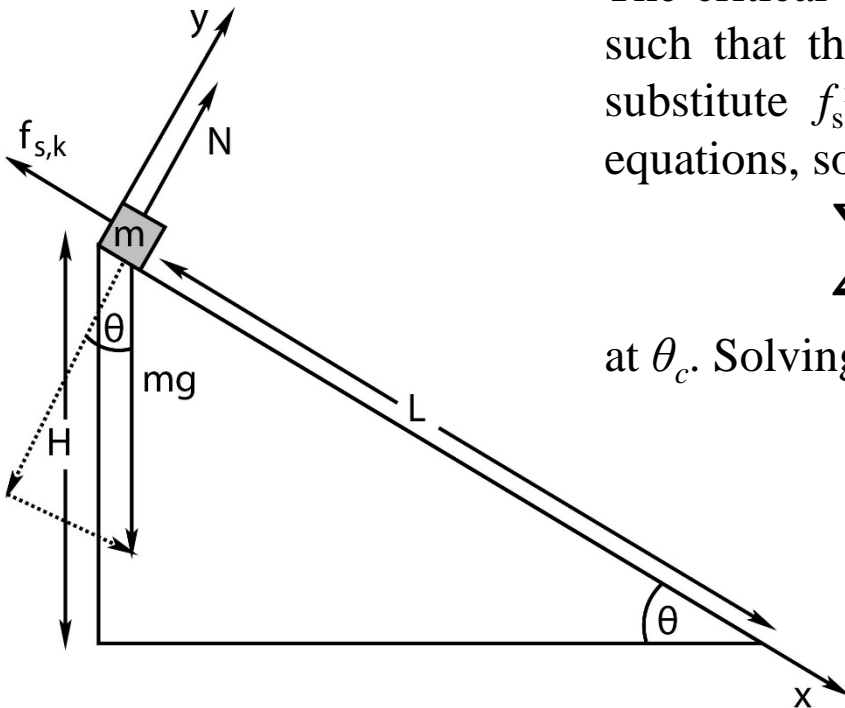


The critical angle is the angle where f_s is as large as it can be such that the block barely doesn't slide. To find it, we can substitute $f_s^{\max} = \mu_s N_c$ where $N_c = mg \cos(\theta_c)$ into both equations, so that the first equation becomes:



Lecture 2. Newton's Laws

Inclined Plane of Length L with Friction



The critical angle is the angle where f_s is as large as it can be such that the block barely doesn't slide. To find it, we can substitute $f_s^{\max} = \mu_s N_c$ where $N_c = mg \cos(\theta_c)$ into both equations, so that the first equation becomes:

$$\sum F_x = mg \sin \theta_c - \mu_s mg \cos \theta_c = 0$$

at θ_c . Solving for θ_c , we get: $\theta_c = \tan^{-1}(\mu_s)$

Once it is moving then the block will accelerate and Newton's Second Law becomes:

$$\sum F_x = mg \sin \theta - \mu_k mg \cos \theta = ma_x$$

which we can solve for the constant acceleration of the block down the incline:

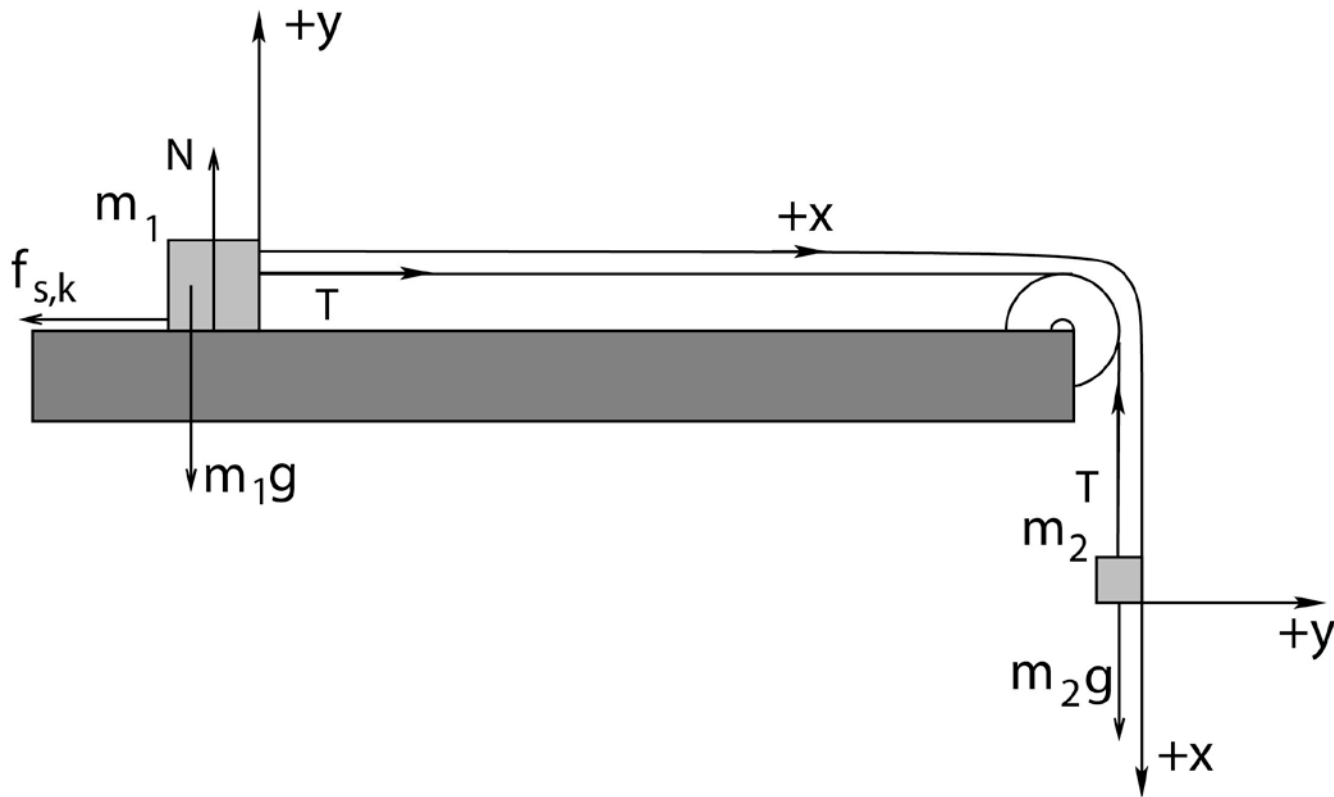
$$a_x = g \sin \theta - \mu_k g \cos \theta = g(\sin \theta - \mu_k \cos \theta)$$

Given a_x , it is now straightforward to answer the second question above. For example, we can integrate twice and find $v_x(t)$ and $x(t)$, use the latter to find the time it takes to reach the bottom, and substitute that time into the former to find the speed at the bottom of the incline.



Lecture 2. Newton's Laws

Block Hanging off of a Table

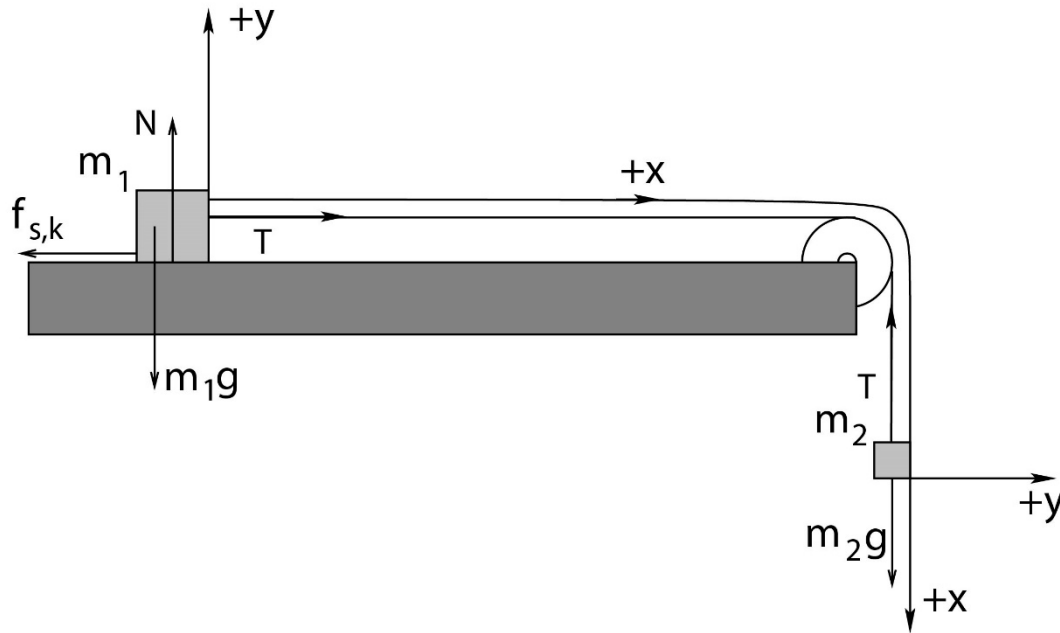


Atwood's machine, sort of, with one block resting on a table with friction and the other dangling over the side being pulled down by gravity near the Earth's surface. Note that we should use an "around the corner" coordinate system as shown, since $a_1 = a_2 = a$ if the string is unstretchable.



Lecture 2. Newton's Laws

Block Hanging off of a Table



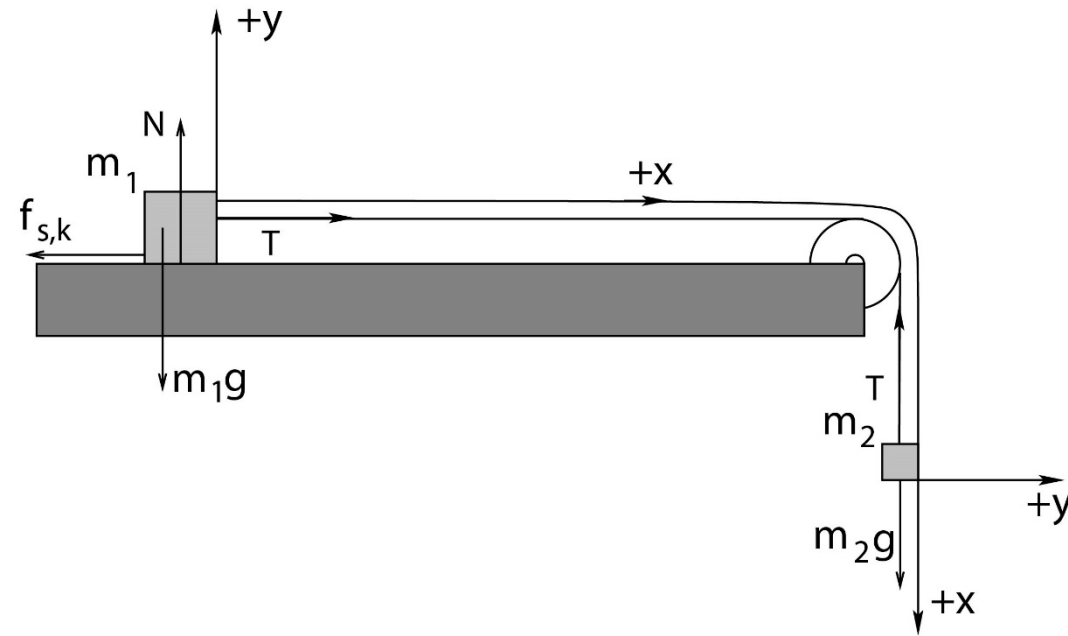
Suppose a block of mass m_1 sits on a table. The coefficients of static and kinetic friction between the block and the table are $\mu_s > \mu_k$ and μ_k respectively. This block is attached by an “ideal” massless unstretchable string running over an “ideal” massless frictionless pulley to a block of mass m_2 hanging off of the table. The blocks are released from rest at time $t = 0$.

What is the largest that m_2 can be before the system starts to move, in terms of the givens and knowns ($m_1, g, \mu_k, \mu_s, \dots$)?



Lecture 2. Newton's Laws

Block Hanging off of a Table



Static force equilibrium ($a_x = a_y = 0$):

$$\sum F_{x1} = T - f_s = 0$$

$$\sum F_{y1} = N - m_1g = 0$$

$$\sum F_{x2} = m_2g - T = 0$$

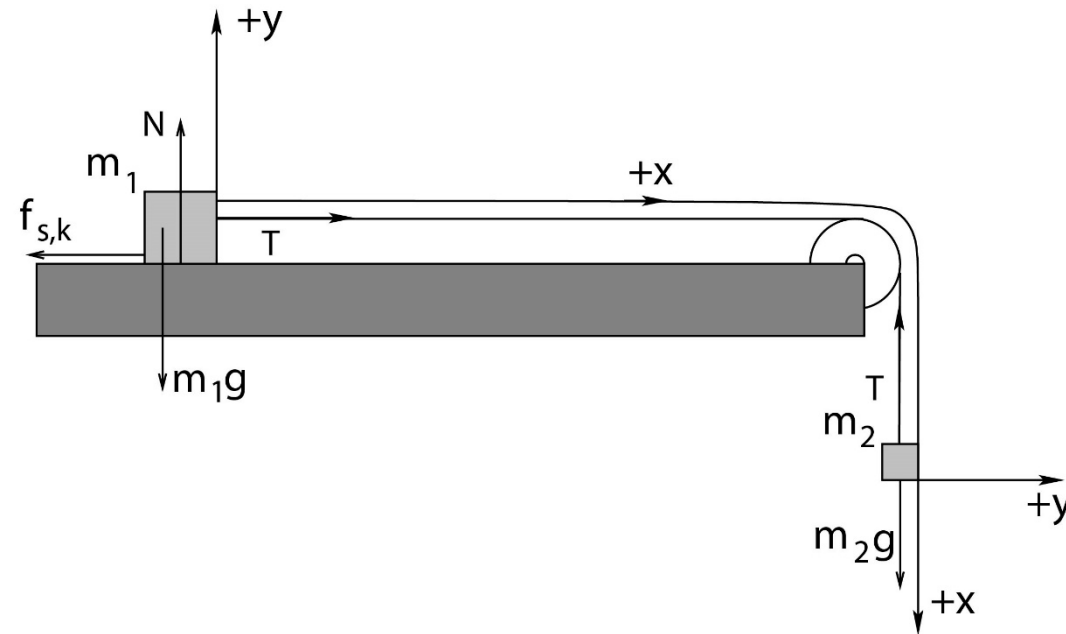
$$\sum F_{y2} = 0$$

From the second equation, $N = m_1g$. At the point where m_2 is the largest it can be (given m_1 and so on) $f_s = f_s^{\max} = \mu_s N = \mu_s m_1g$. If we substitute this in and add the two x equations, the T cancels and we get: $m_2^{\max} g - \mu_s m_1g = 0$ Thus: $m_2^{\max} = \mu_s m_1$



Lecture 2. Newton's Laws

Block Hanging off of a Table



If m_2 is larger than this minimum, so m_1 will slide to the right as m_2 falls. We will have to solve Newton's Second Law for both masses in order to obtain the non-zero acceleration to the right and down, respectively:

$$\sum F_{x1} = T - f_k = m_1 a$$

$$\sum F_{y1} = N - m_1 g = 0$$

$$\sum F_{x2} = m_2 g - T = m_2 a$$

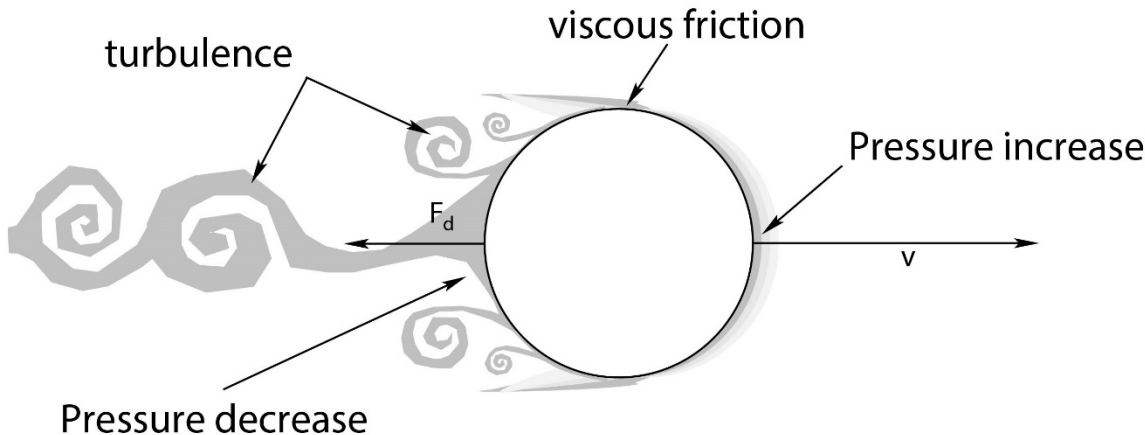
$$\sum F_{y2} = 0$$

If we substitute the fixed value for $f_k = \mu_k N = \mu_k m_1 g$ and then add the two x equations once again (using the fact that both masses have the same acceleration because the string is unstretchable as noted in our original construction of round-the-corner coordinates), the tension T cancels and we get:

$$m_2 g - \mu_s m_1 g = (m_1 + m_2) a \text{ or } a = \frac{m_2 g - \mu_s m_1 g}{(m_1 + m_2)}$$

Lecture 2. Newton's Laws

Drag Forces



A “cartoon” illustrating the differential force on an object moving through a fluid.

When the object is moving with respect to the fluid then we empirically observe that a friction-like force is exerted on the object called **drag**.

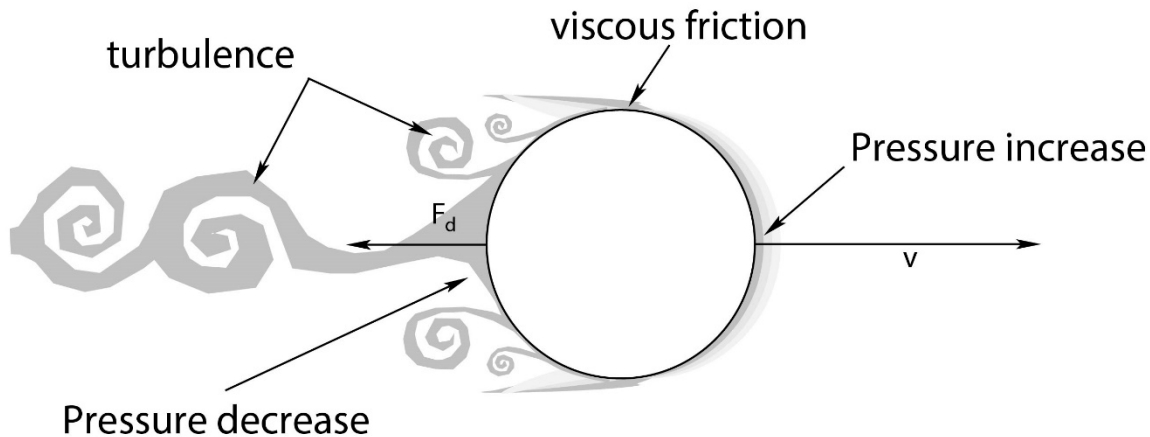
Drag Force is the “frictional” force exerted by a fluid (liquid or gas) on an object that moves through it. Like kinetic friction, it always opposes the direction of *relative* motion of the object and the medium

Note well: When an object is elongated and passes through a fluid parallel to its long axis with a comparatively small forward-facing cross section compared to its total area, we say that it is a streamlined object as the fluid tends to pass over it in laminar flow. A streamlined object will often have its total drag dominated by skin friction. A bluff object, in contrast has a comparatively large cross-sectional surface facing forward and will usually have the total drag dominated by form drag.



Lecture 2. Newton's Laws

Drag Forces



Note well:

When an object is elongated and passes through a fluid parallel to its long axis with a comparatively small forward-facing cross section compared to its total area, we say that it is a ***streamlined object*** as the fluid tends to pass over it in laminar flow. A streamlined object will often have its total drag dominated by skin friction.

A ***bluff object***, in contrast has a comparatively large cross-sectional surface facing forward and will usually have the total drag dominated by form drag.



Lecture 2. Newton's Laws

Drag Forces

Drag is an extremely complicated force. It depends on a vast array of things including but not limited to:

- The size of the object.
- The shape of the object.
- The relative velocity of the object through the fluid.
- The state of the fluid (e.g. its velocity field including any internal turbulence).
- The density of the fluid.
- The viscosity of the fluid (we will learn what this is later).
- The properties and chemistry of the surface of the object (smooth versus rough, strong or weak chemical interaction with the fluid at the molecular level).
- The orientation of the object as it moves through the fluid, which may be fixed in time (streamlined versus bluff motion) or varying in time (as, for example, an irregularly shaped object tumbles).

To eliminate most of this complexity and end up with “force rules” that will often be quantitatively predictive we will use a number of idealizations. We will only consider smooth, uniform, nonreactive surfaces of convex bluff objects (like spheres) or streamlined objects (like rockets or arrows) moving through uniform, stationary fluids where we can ignore or treat separately the other non-drag (e.g. buoyant) forces acting on the object.



Lecture 2. Newton's Laws

Drag Forces

There are two dominant contributions to drag for objects of this sort.

The first, as noted above, is *form drag* – the difference in pressure times projective area between the front of an object and the rear of an object. It is strongly dependent on both the shape and orientation of an object and requires at least some turbulence in the trailing wake in order to occur.

The second is *skin friction*, the friction-like force resulting from the fluid rubbing across the skin at right angles in laminar flow.



Lecture 2. Newton's Laws

Stokes, or Laminar Drag

The first is when the object is moving through the fluid relatively slowly and/or is arrow-shaped or rocket-ship-shaped so that streamlined **laminar** drag (skin friction) is dominant. In this case there is relatively little form drag, and in particular, there is little or no **turbulence** – eddies of fluid spinning around an axis – in the wake of the object as the presence of turbulence (which we will discuss in more detail later when we consider fluid dynamics) breaks up laminar flow.

This “low-velocity, streamlined” skin friction drag is technically named **Stokes’ drag** or laminar drag and has the idealized force rule:

$$\vec{F}_d = -b\vec{v}$$

This is the simplest sort of drag – a drag force *directly proportional to the velocity of relative motion of the object through the fluid and oppositely directed*.

Stokes derived the following relation for the dimensioned number b_l (the laminar drag coefficient) that appears in this equation for a sphere of radius R :

$$b_l = -6\pi\mu R$$

where μ is the dynamical viscosity.



Lecture 2. Newton's Laws

Rayleigh, or Turbulent Drag

On the other hand, if one moves an object through a fluid *too* fast – where the actual speed depends in detail on the actual size and shape of the object, how bluff or streamlined it is – pressure builds up on the leading surface and *turbulence* appears in its trailing wake in the fluid.

This high velocity, *turbulent drag* exerts a force described by a quadratic dependence on the relative velocity due to Lord Rayleigh:

$$\vec{F}_d = -\frac{1}{2}\rho C_d A |v| \vec{v} = -b_t |v| \vec{v}$$

It is still *directed opposite to the relative velocity of the object and the fluid* but now is proportional to that velocity *squared*. In this formula ρ is the density of the fluid through which the object moves (so denser fluids exert more drag as one would expect) and A is the cross-sectional area of the object perpendicular to the direction of motion, also known as the orthographic projection of the object on any plane perpendicular to the motion. For example, for a sphere of radius R , the orthographic projection is a circle of radius R and the area $A = \pi R^2$.

The number C_d is called the drag coefficient and is a dimensionless number that depends on relative speed, flow direction, object position, object size, fluid viscosity and fluid density.



Lecture 2. Newton's Laws

Example: Falling From a Plane and Surviving

Suppose you fall from a large height (long enough to reach terminal velocity) to hit a haystack of height H that exerts a nice, uniform force to slow you down all the way to the ground, smoothly compressing under you as you fall. In that case, your initial velocity at the top is v_t , down. In order to stop you before $y = 0$ (the ground) you have to have a net acceleration $-a$ such that:

$$\begin{aligned}v(t_g) &= 0 = v_t - at_g \\ y(t_g) &= 0 = H - v_t t_g - \frac{1}{2} at_g^2\end{aligned}$$

If we solve the first equation for t_g and substitute it into the second and solve for the magnitude of a , we will get:

$$-v_t^2 = -2aH \quad \text{or} \quad a = \frac{v_t^2}{2H}$$

We know also that

$$F_{\text{haystack}} - mg = ma \quad \text{or}$$

$$F_{\text{haystack}} = ma + mg = m(a + g) = mg' = m\left(\frac{v_t^2}{2H} + g\right)$$



Lecture 2. Newton's Laws

Example: Falling From a Plane and Surviving

$$F_{haystack} = ma + mg = m(a + g) = mg' = m\left(\frac{v_t^2}{2H} + g\right)$$

Let's suppose the haystack was $H = 1.25$ meter high and, because you cleverly landed on it in a “bluff” position to keep v_t as small as possible, you start at the top moving at only $v_t = 50$ meters per second. Then $g' = a + g$ is approximately 1009.8 meters/second², 103 ‘gees’, and the force the haystack must exert on you is 103 times your normal weight. You actually have a small chance of surviving this stopping force, but it isn't a very large one.

To have a better chance of surviving, one needs to keep the g-force under 100, ideally well under 100. Since the “haystack” portion of the acceleration needed is inversely proportional to H we can see that a 10 meter haystack would lead to 13.5 gees



Lecture 3. Work and Energy

Work and Kinetic Energy

If you integrate a constant acceleration of an object twice, you obtain:

$$v(t) = at + v_0$$
$$x(t) = \frac{1}{2}at^2 + v_0t + x_0$$

where v_0 is the initial speed and x_0 is the initial x position at time $t = 0$.

Now, suppose you want to find the speed v_1 the object will have when it reaches position x_1 .

One can algebraically, once and for all note that this must occur at some time t_1 such that:

$$v(t_1) = at_1 + v_0 = v_1$$
$$x(t_1) = \frac{1}{2}at_1^2 + v_0t_1 + x_0 = x_1$$

We can algebraically solve the first equation once and for all for t_1 :

$$t_1 = \frac{v_1 - v_0}{a}$$

and substitute the result into the second equation, eliminating time altogether from the solutions:



Lecture 3. Work and Energy

Work and Kinetic Energy

$$\begin{aligned}\frac{1}{2}a\left(\frac{v_1 - v_0}{a}\right)^2 + v_0\left(\frac{v_1 - v_0}{a}\right) + x_0 &= x_1 \\ \frac{1}{2}a(v_1^2 - 2v_0v_1 + v_0^2) + \left(\frac{v_0v_1 - v_0^2}{a}\right) &= x_1 - x_0 \\ v_1^2 - 2v_0v_1 + v_0^2 + 2v_0v_1 - v_0^2 &= 2a(x_1 - x_0) \\ \text{or } v_1^2 - v_0^2 &= 2a(x_1 - x_0)\end{aligned}$$

Lets consider a constant acceleration in one dimension only:

$$v_1^2 - v_0^2 = 2a\Delta x$$

If we multiply by m (the mass of the object) and move the annoying 2 over to the other side, we can make the constant acceleration a into a constant force $F_x = ma$:

$$\begin{aligned}(ma)\Delta x &= \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 \\ F_x\Delta x &= \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2\end{aligned}$$

We now define the **work done by the constant force F_x on the mass m** as it moves through the distance Δx to be: $\Delta W = F_x\Delta x$

Work is a form of energy.

$$1 \text{ Joule} = 1 \text{ Newton} \cdot \text{meter} = 1 \frac{\text{kilogram} \cdot \text{meter}^2}{\text{second}^2}$$



Lecture 3. Work and Energy

Kinetic Energy

Let's define the quantity changed by the work to be the kinetic energy and will use the symbol K to represent it in this work:

$$K = \frac{1}{2}mv^2$$

Work-Kinetic Energy Theorem:

The work done on a mass by the total force acting on it is equal to the change in its kinetic energy.

and as an equation that is correct for constant one dimensional forces only:

$$\Delta W = F_x \Delta x = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \Delta K$$



Lecture 3. Work and Energy

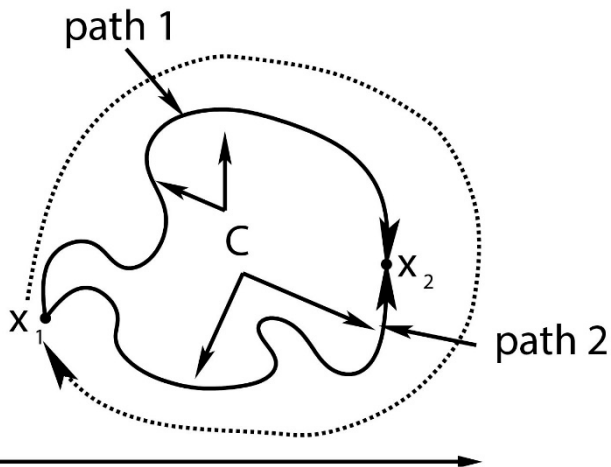
Conservative Forces: Potential Energy

We define a **conservative force** to be one such that the work done by the force as you move a point mass from point \vec{x}_1 to point \vec{x}_2 is independent of the path used to move between the points:

$$W_{loop} = \oint_{\vec{x}_1(\text{path 1})}^{\vec{x}_2} \vec{F} \cdot d\vec{l} = \oint_{\vec{x}_1(\text{path 2})}^{\vec{x}_2} \vec{F} \cdot d\vec{l}$$

In this case (only), the work done going around an arbitrary closed path (starting and ending on the same point) will be identically zero!

$$W_{loop} = \oint_C \vec{F} \cdot d\vec{l} = 0$$



The work done going around an arbitrary loop by a conservative force is zero. This ensures that the work done going between two points is *independent* of the path taken, its defining characteristic.



Lecture 3. Work and Energy

Conservative Forces: Potential Energy

Since the work done moving a mass m from an arbitrary starting point to any point in space is the **same** independent of the path, we can assign each point in space a numerical value: the work done by *us* on mass m , *against* the conservative force, to reach it.

This is the *negative* of the work done by the force. We do it with this sign for reasons that will become clear in a moment. We call this function the ***potential energy*** of the mass m associated with the conservative force \vec{F} :

$$U(\vec{x}) = - \int_{x_0}^x \vec{F} \cdot d\vec{x} = -W$$

Note Well: that only one limit of integration depends on x ; the other depends on where you choose to make the potential energy zero. This is a *free choice*. No physical result that can be measured or observed can uniquely depend on where you choose the potential energy to be zero.



Lecture 3. Work and Energy

Conservation of Mechanical Energy

The principle of the *Conservation of Mechanical Energy*:

The total mechanical energy (defined as the sum of its potential and kinetic energies) of a particle being acted on by only conservative forces is constant.

Or, if only conservative forces act on an object and U is the potential energy function for the total conservative force, then

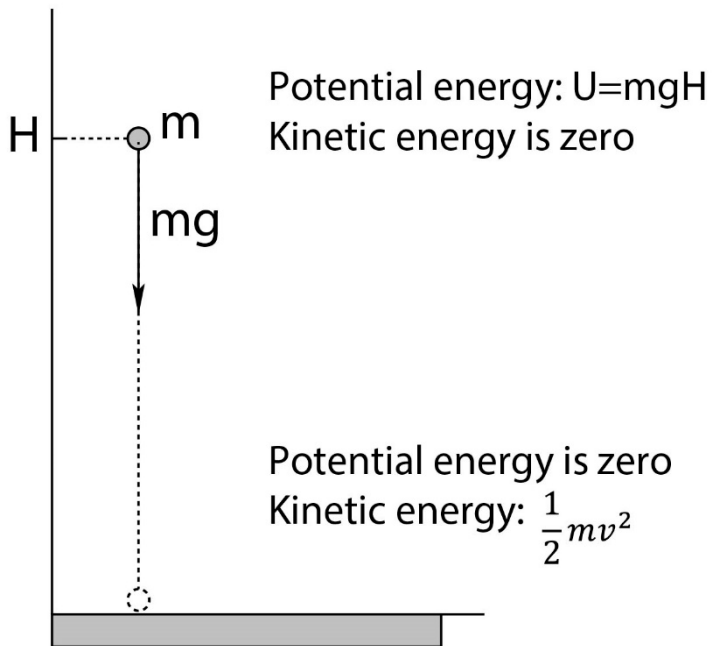
$$E_{mech} = K + U = A \quad \text{scalar constant}$$

The fact that the force is the negative derivative of the potential energy of an object *means* that *the force points in the direction the potential energy decreases in*.



Lecture 3. Work and Energy

Example: Falling Ball Reprise



To see how powerful this is, let us look back at a falling object of mass m (neglecting drag and friction). First, we have to determine the gravitational potential energy of the object a height y above the ground (where we will choose to set $U(0) = 0$):

$$U(y) = - \int_0^y (-mg) dy = mgy$$

Now, suppose we have our ball of mass m at the height H and drop it from rest. How fast is it going when it hits the ground? This time we simply write the total energy of the ball at the top (where the potential is mgH and the kinetic is zero) and the bottom (where the potential is zero and kinetic is $\frac{1}{2}mv^2$ and set the two equal! Solve for v , done:

$$E_i = mgH + (0) = (0) + \frac{1}{2}mv^2 = E_f$$

$$\text{or } v = \sqrt{2gH}$$



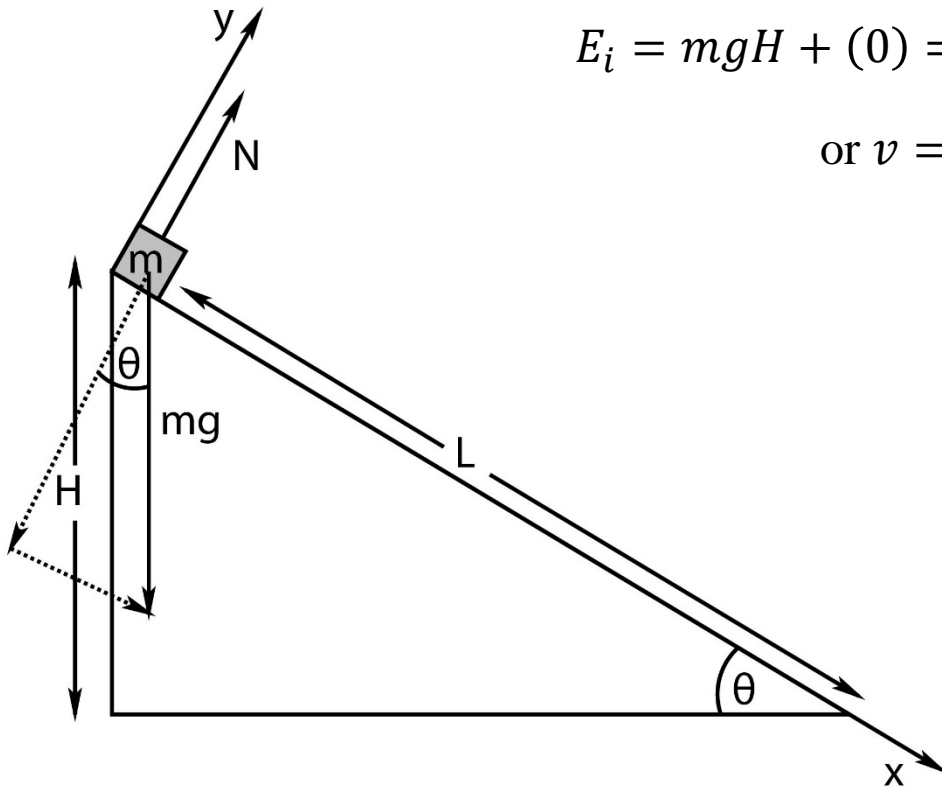
Lecture 3. Work and Energy

Example: Block Sliding Down Frictionless Incline Reprise

The block starts out a height H above the ground, with potential energy mgH and kinetic energy of 0 . It slides to the ground (no non-conservative friction!) and arrives with no potential energy and kinetic energy $\frac{1}{2}mv^2$

$$E_i = mgH + (0) = (0) + \frac{1}{2}mv^2 = E_f$$

$$\text{or } v = \sqrt{2gH}$$

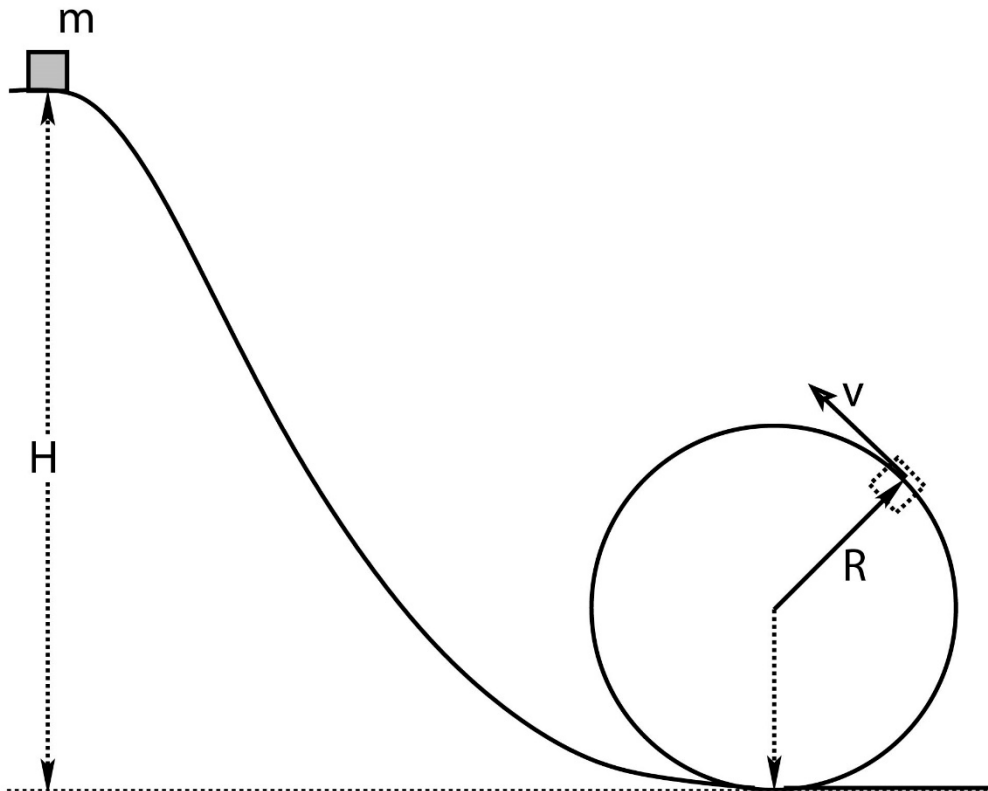




Lecture 3. Work and Energy

Example: Looping the Loop

What is the minimum height H such that a block of mass m loops-the-loop (stays *on the frictionless track all the way around the circle*)?





Lecture 3. Work and Energy

Example: Looping the Loop

Here we need two physical principles: *Newton's Second Law* and *the kinematics of circular motion* since the mass is undoubtedly moving in a circle if it stays on the track. Here's the way we reason:

“If the block moves in a circle of radius R at speed v , then its acceleration towards the center must be $a_c = v^2/R$. Newton's Second Law then tells us that the *total force component* in the direction of the center must be mv^2/R . That force can only be made out of (a component of) gravity and the normal force, which points towards the center. So we can relate the normal force to the speed of the block on the circle at any point.”

At the top (where we expect v to be at its minimum value, assuming it stays on the circle) gravity points straight towards the center of the circle of motion, so we get:

$$mg + N = \frac{mv^2}{R}$$

and in the limit that $N \rightarrow 0$ (“barely” looping the loop) we get the condition:

$$mg = \frac{mv_t^2}{R}$$

where v_t is the (minimum) speed at the top of the track needed to loop the loop.



Lecture 3. Work and Energy

Example: Looping the Loop

Now we need to relate the speed at the top of the circle to the original height H it began at. This is where we need our third principle – *Conservation of Mechanical Energy*!

With energy we don't care about the shape of the track, only that the track do no work on the mass which (since it is frictionless and normal forces do no work) is in the bag. Thus:

$$E_i = mgH = mg2R + \frac{1}{2}mv_t^2 = E_f$$

If you put these two equations together (e.g. solve the first for mv_t^2 and substitute it into the second, then solve for H in terms of R) you should get

$$H_{min} = 5R/2.$$



Lecture 3. Work and Energy

Example: Generalized Work-Mechanical Energy Theorem

Let's consider what happens if **both** conservative and nonconservative forces are acting **on a particle**. In that case the argument above becomes:

$$W_{rot} = W_C + W_{NC} = \Delta K$$

$$\text{or } W_{NC} = \Delta K - W_C = \Delta K + \Delta U = \Delta E_{mech}$$

which we state as the **Generalized Non-Conservative Work-Mechanical Energy Theorem**:

The work done by all the non-conservative forces acting on a particle equals the change in its total mechanical energy.



Lecture 3. Work and Energy

Example: Heat and Conservation of Energy

The important empirical law is the ***Law of Conservation of Energy***. Whenever we examine a physical system and try very hard to keep track of all of the mechanical energy exchanges within that system and between the system and its surroundings, we find that we can always account for them all without any gain or loss.

In other words, we find that the total mechanical energy of an ***isolated*** system never changes, and if we add or remove mechanical energy to/from the system, it has to come from or go to somewhere outside of the system. This result, applied to well defined systems of particles, can be formulated as the ***First Law of Thermodynamics***:

$$\Delta Q_{in} = \Delta E_{of} + W_{by}$$

In words, the heat energy flowing *in* to a system equals the change in the internal total mechanical energy *of* the system plus the external work (if any) done *by* the system on its surroundings.



Lecture 3. Work and Energy

Example: Heat and Conservation of Energy

When a block slides down a rough table from some initial velocity to rest, kinetic friction turns the bulk organized kinetic energy of the collectively moving mass into *disorganized microscopic energy* – heat.

As the rough microscopic surfaces bounce off of one another and form and break chemical bonds, it sets the actual molecules of the block bounding, increasing the internal microscopic mechanical energy of the block and *warming it up*.



Lecture 3. Work and Energy

Power

The energy in a given system is not, of course, usually constant in time. Energy is added to a given mass, or taken away, at some rate.

There are many times when we are given the **rate** at which energy is added or removed in time, and need to find the total energy added or removed. This rate is called the **power**.

Power: The rate at which work is done, or energy released into a system.

$$dW = \vec{F} d\vec{x} = \vec{F} \cdot \frac{d\vec{x}}{dt} dt$$

$$P = \frac{dW}{dt} = \vec{F} \cdot \vec{v}$$

$$\text{so that } \Delta W = \Delta E_{tot} = \int P dt$$

The units of power are clearly Joules/sec = Watts. Another common unit of power is “Horsepower”, 1 HP = 746 W.



Lecture 3. Work and Energy

Equilibrium

The **force** is given by the **negative gradient of the potential energy**:

$$\vec{F} = -\vec{\nabla} U$$

or (in each direction): $F_x = -\frac{dU}{dx}$, $F_y = -\frac{dU}{dy}$, $F_z = -\frac{dU}{dz}$,

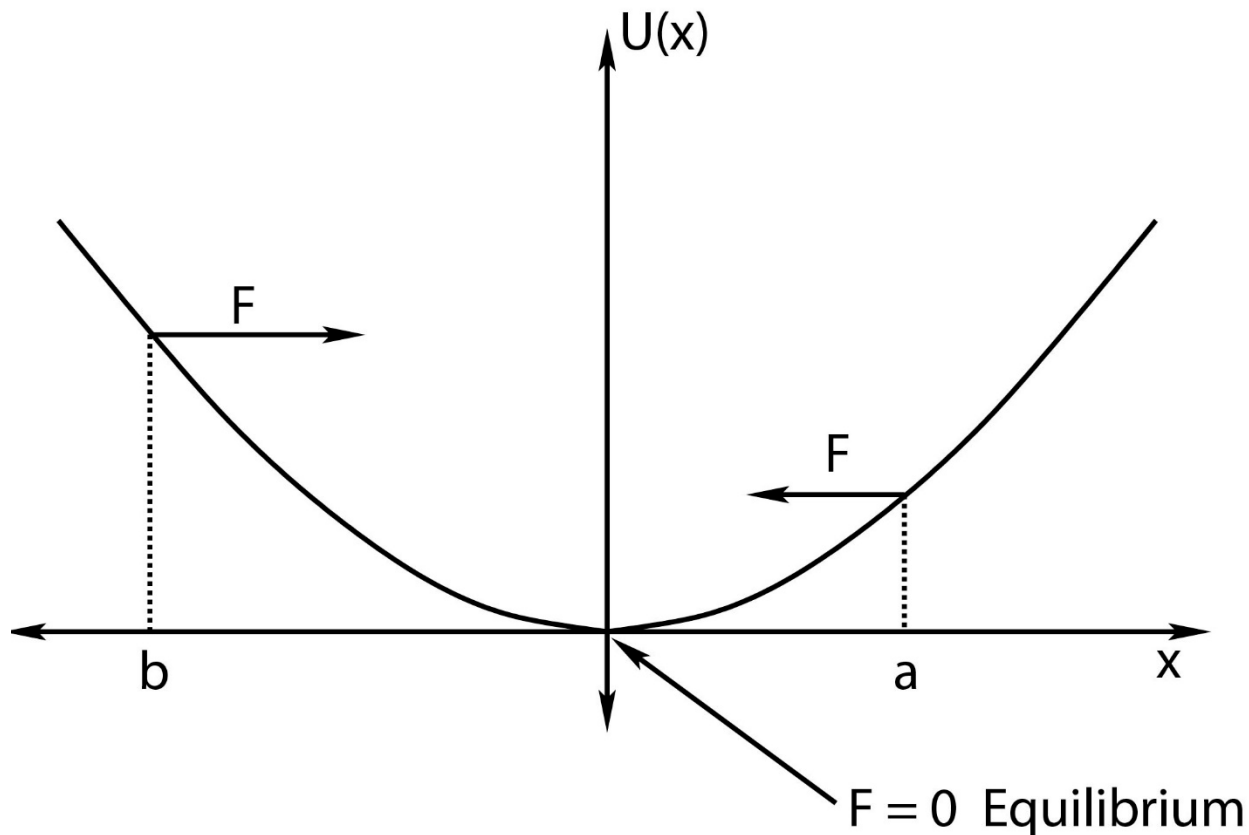
or the force is the negative **slope** of the potential energy function in this direction.

The *meaning* of this is that if a particle moves in the direction of the (conservative) force, it speeds up. If it speeds up, its kinetic energy increases. If its kinetic energy increases, its potential energy must *decrease*. The force (component) acting on a particle is thus the rate *at which the potential energy* decreases (the negative slope) in any given direction



Lecture 3. Work and Energy

Equilibrium

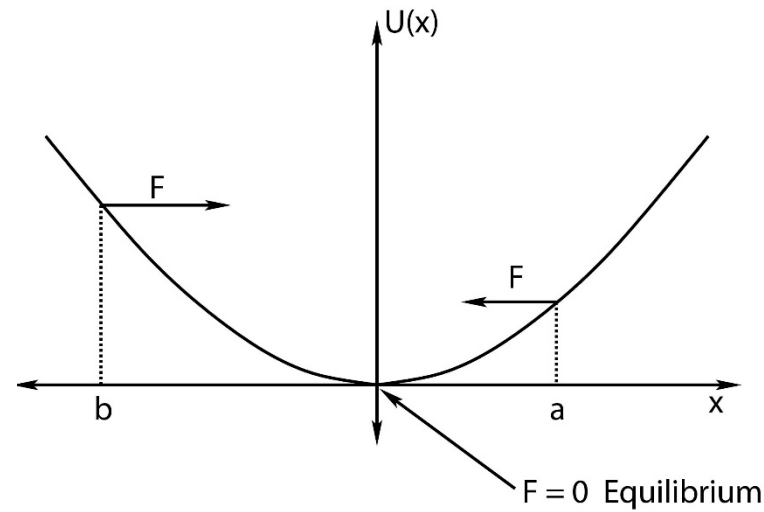


A one-dimensional potential energy curve $U(x)$.



Lecture 3. Work and Energy

Equilibrium



A one-dimensional potential energy curve $U(x)$.

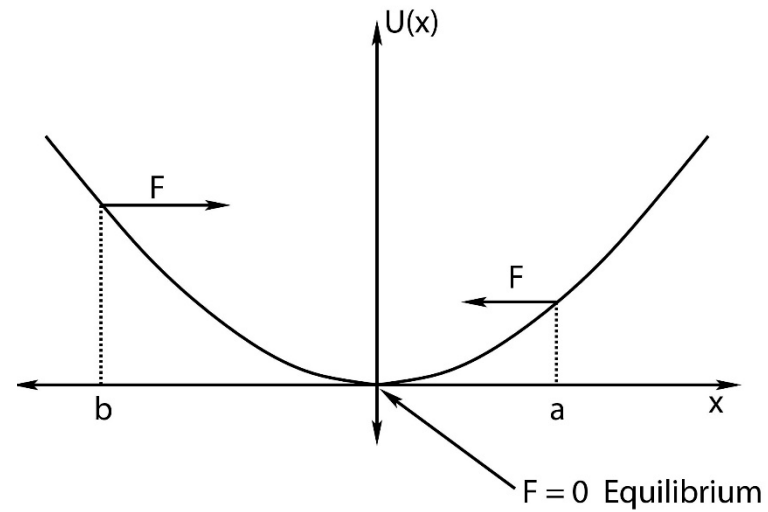
At the point labelled a , the x -slope of $U(x)$ is *positive*. The x (component of the) force, therefore, is in the negative x direction. At the point b , the x -slope is *negative* and the force is correspondingly positive. Note well that the force gets larger as the slope of $U(x)$ gets larger (in magnitude).

The point in the middle, at $x = 0$, is *special*. Note that this is a *minimum* of $U(x)$ and hence the x -slope is zero. Therefore the x -directed force F at that point is zero as well. A point at which the force on an object is zero is, as we previously noted, a point of ***static force equilibrium*** – a particle placed there at rest will remain there at rest.



Lecture 3. Work and Energy

Equilibrium



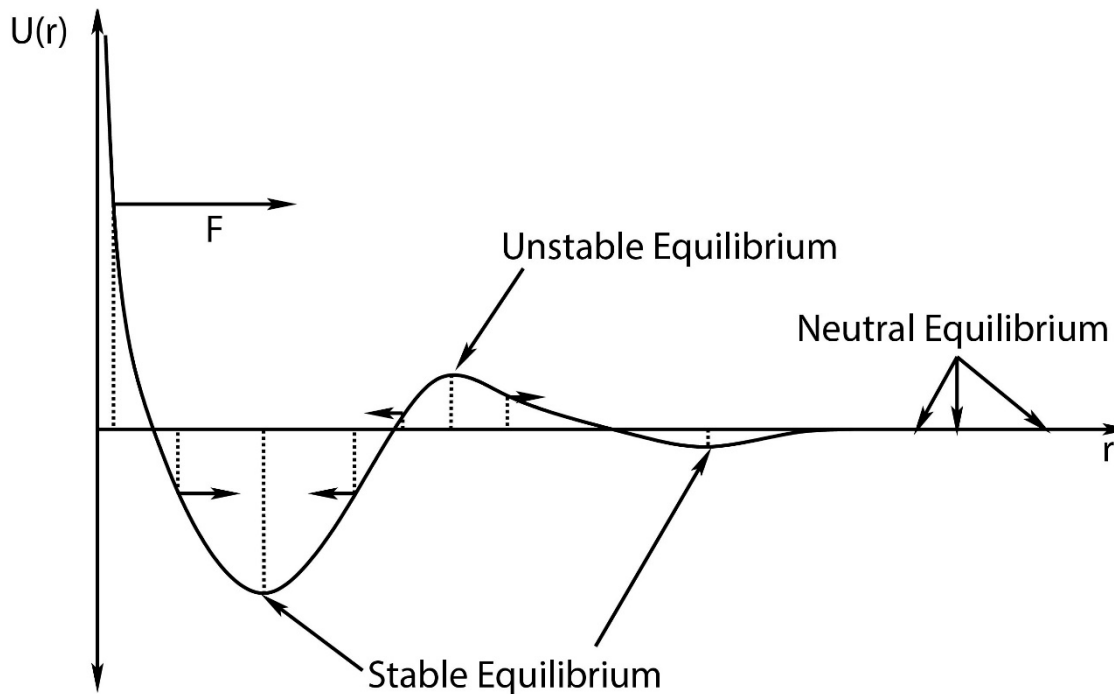
A one-dimensional potential energy curve $U(x)$.

In this particular figure, if one moves the particle a small distance to the right or the left of the equilibrium point, the force *pushes the particle back towards equilibrium*. Points where the force is zero and small displacements cause a restoring force in this way are called *stable equilibrium points*. As you can see, the ***isolated minima*** of a potential energy curve (or surface, in higher dimensions) are all ***stable equilibria***.



Lecture 3. Work and Energy

Equilibrium

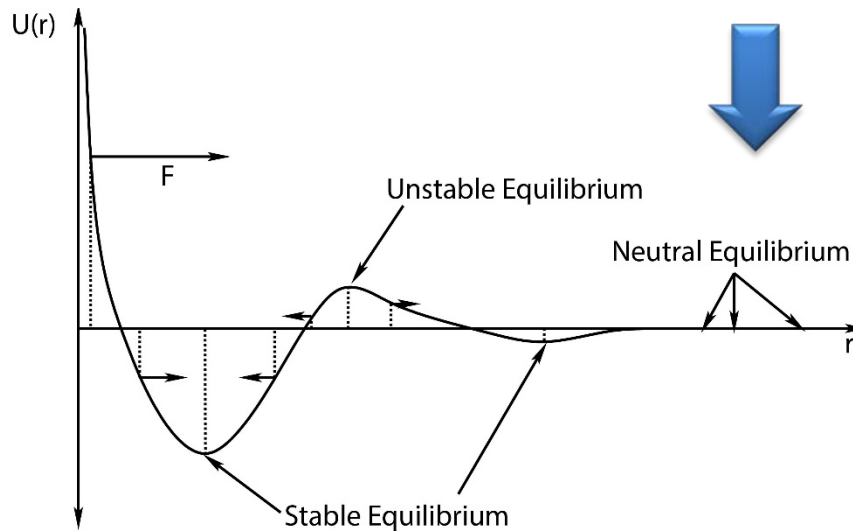


A fairly generic potential energy shape for microscopic (atomic or molecular) interactions, drawn to help exhibits features one might see in such a curve more than as a realistically scaled potential energy in some set of units. In particular, the curve exhibits stable, unstable, and neutral equilibria for a radial potential energy as a function of r , the distance between two e.g. atoms.



Lecture 3. Work and Energy

Equilibrium

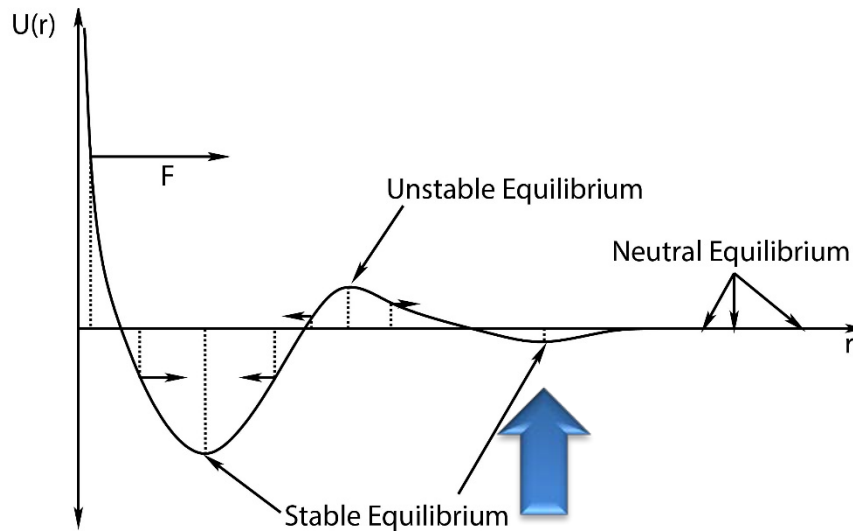


At very long ranges, the forces between neutral atoms are extremely small, effectively *zero*. This is illustrated as an extended region where the potential energy is *flat* for large r . Such a range is called **neutral equilibrium** because there are no forces that either restore or repel the two atoms. Neutral equilibrium is *not stable* in the specific sense that a particle placed there with *any nonzero velocity* will move freely (according to Newton's First Law). Since it is nearly impossible to prepare an atom at absolute rest relative to another particle, one basically “never” sees two unbound microscopic atoms with a large, perfectly constant spatial orientation.



Lecture 3. Work and Energy

Equilibrium

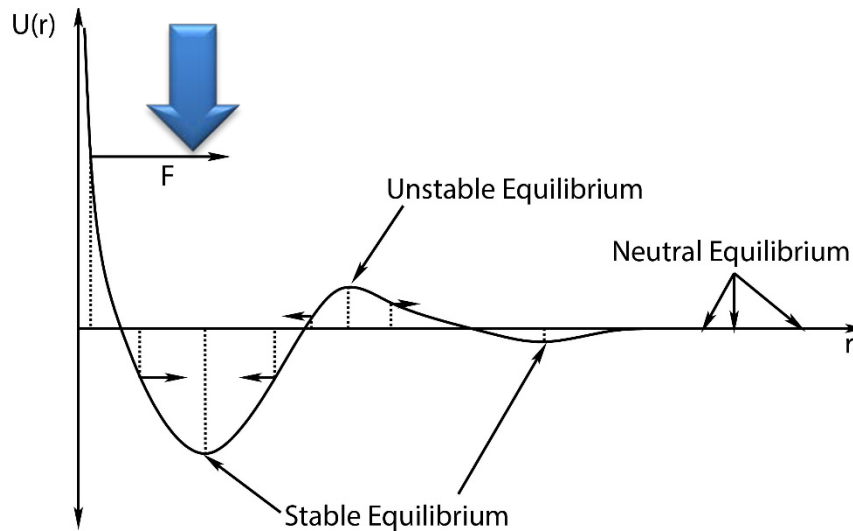


As the two atoms near one another, their interaction becomes first *weakly attractive* due to e.g. quantum dipole-induced dipole interactions and then *weakly repulsive* as the two atoms start to “touch” each other. There is a potential energy minimum in between where two atoms separated by a certain distance can be in stable equilibrium without being chemically bound.



Lecture 3. Work and Energy

Equilibrium

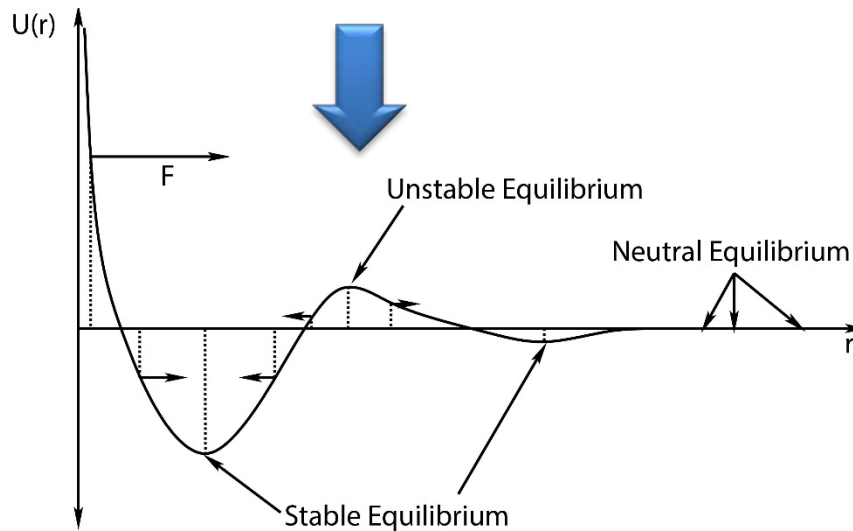


Atoms that approach one another still more closely encounter a second potential energy well that is at first strongly attractive followed by a hard core repulsion as the electron clouds are prevented from interpenetrating by e.g. the Pauli exclusion principle. This second potential energy well is often modelled by a Lennard-Jones potential energy. It also has a point of stable equilibrium.



Lecture 3. Work and Energy

Equilibrium

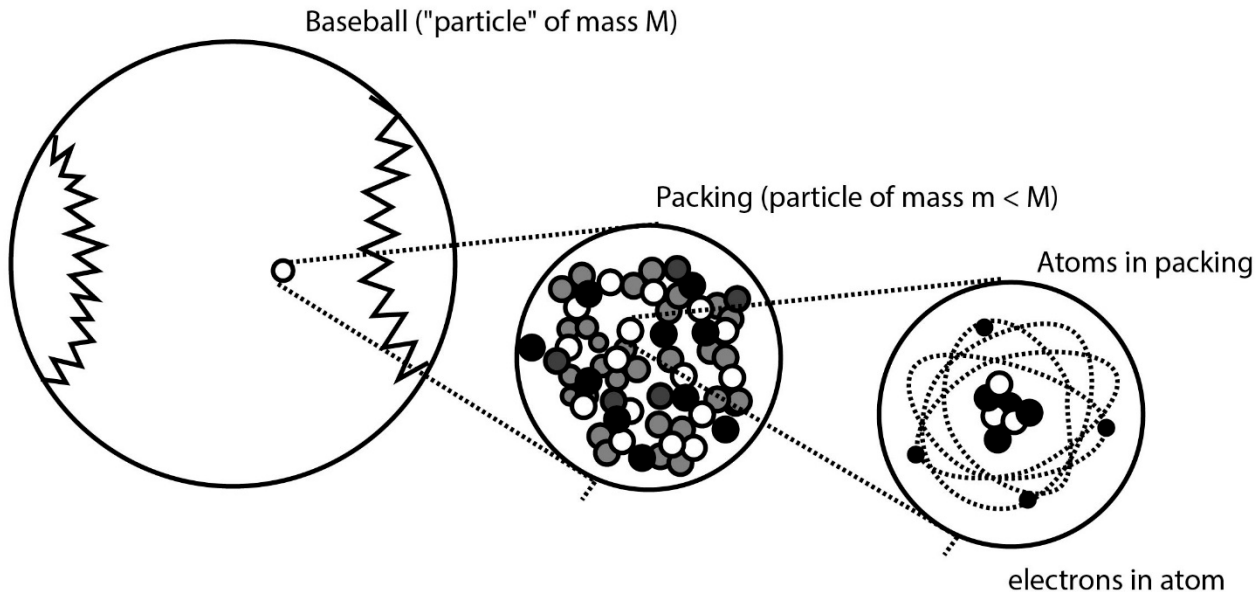


In between, there is a point where the growing attraction of the inner potential energy well and the growing repulsion of the outer potential energy well *balance*, so that the potential energy function has a *maximum*. At this maximum the slope is zero (so it is a position of force equilibrium) but because the force on either side of this point pushes the particle *away* from it, this is a point of ***unstable equilibrium***. Unstable equilibria occur at ***isolated maxima*** in the potential energy function, just as stable equilibria occur at *isolated minima*.



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Systems of Particles



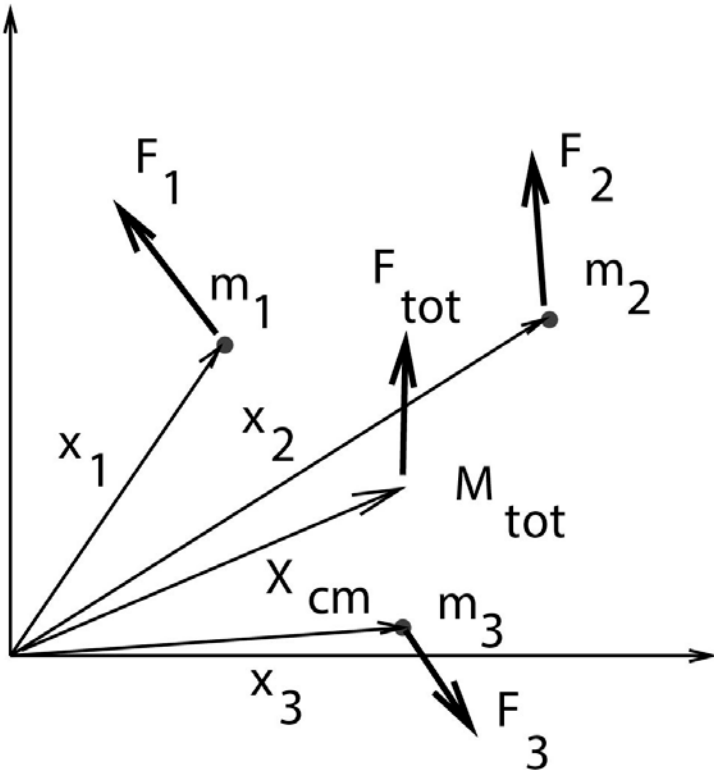
An object such as a baseball is not really a particle. It is made of *many, many* particles – even the atoms it is made of are made of many particles *each*. Yet it *behaves* like a particle as far as Newton's Laws are concerned.

We will obtain this collective behavior by **averaging**, or *summing* over at successively larger scales, the physics that we know applies at the smallest scale to things that *really* are particles.



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Newton's Laws for a System of Particles – Center of Mass



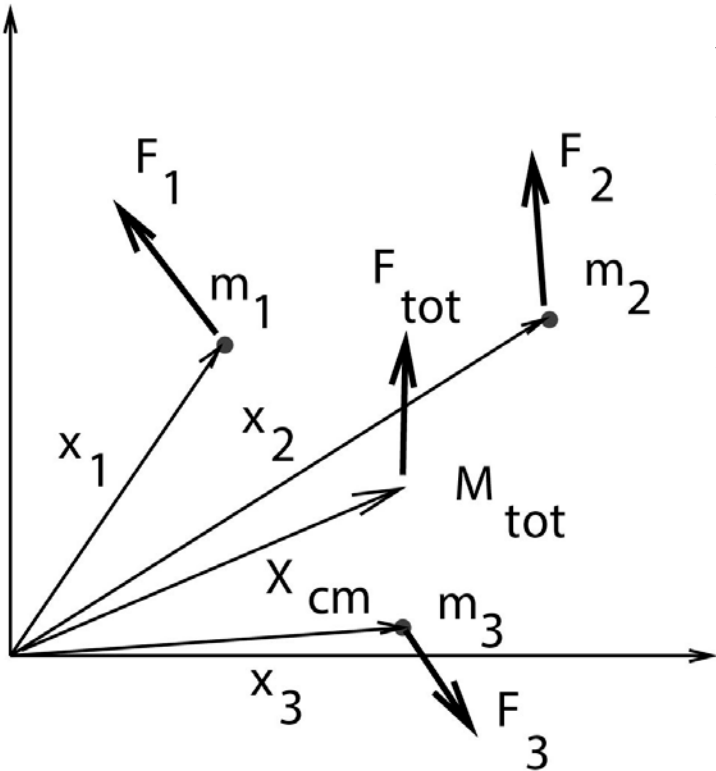
A system of $N = 3$ particles is shown, with various forces \vec{F}_i acting on the masses (which therefore each have their own accelerations \vec{a}_i). From this, we construct a *weighted average* acceleration of the system, in such a way that Newton's Second Law is satisfied for the *total* mass.

Suppose we have a system of N particles, each of which is experiencing a force. Some (part) of those forces are “external” – they come from outside of the system. Some (part) of them may be “internal” – equal and opposite force pairs between particles that help hold the system together (solid) or allow its component parts to interact (liquid or gas).



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Newton's Laws for a System of Particles – Center of Mass



We would like the total force to act on the total mass of this system as if it were a “particle”. That is, we would like for:

$$\vec{F}_{tot} = M_{tot}\vec{A}$$

where \vec{A} is the “acceleration of the system”. Newton’s Second Law for a system of particles is written as:

$$\begin{aligned}\vec{F}_{tot} &= \sum_i \vec{F}_i = \sum_i m_i \frac{d^2 \vec{x}_i}{dt^2} = \\ &= \left(\sum_i m_i \right) \frac{d^2 \vec{X}}{dt^2} = M_{tot} \frac{d^2 \vec{X}}{dt^2} = M_{tot} \vec{A}\end{aligned}$$



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Newton's Laws for a System of Particles – Center of Mass

$$\sum_i m_i \frac{d^2 \vec{x}_i}{dt^2} = M_{tot} \frac{d^2 \vec{X}}{dt^2}$$

Basically, if we define an \vec{X} such that this relation is true then Newton's second law is recovered for the entire system of particles “located at \vec{X} ” as if that location were indeed a particle of mass M_{tot} itself. We can rearrange this a bit as:

$$\frac{d\vec{V}}{dt} = \frac{d^2 \vec{X}}{dt^2} = \frac{1}{M_{tot}} \sum_i m_i \frac{d^2 \vec{x}_i}{dt^2} = \frac{1}{M_{tot}} \sum_i m_i \frac{d\vec{v}_i}{dt}$$

and can integrate twice on both sides. The first integral is:

$$\frac{d\vec{X}}{dt} = \vec{V} = \frac{1}{M_{tot}} \sum_i m_i \vec{v}_i + \vec{V}_0 = \frac{1}{M_{tot}} \sum_i m_i \frac{d\vec{x}_i}{dt} + \vec{V}_0$$

and the second is:
$$\vec{X} = \frac{1}{M_{tot}} \sum_i m_i \vec{x}_i + \vec{V}_0 t + \vec{X}_0$$



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Newton's Laws for a System of Particles – Center of Mass

We define the position of **the center of mass** to be:

$$M\vec{X}_{\text{cm}} = \sum_i m_i \vec{x}_i \quad \text{or} \quad \vec{X}_{\text{cm}} = \frac{1}{M} \sum_i m_i \vec{x}_i$$

Not all systems we treat will appear to be made up of point particles. Most solid objects or fluids appear to be made up of a *continuum* of mass, a *mass distribution*. In this case we need to do the sum by means of *integration*, and our definition becomes:

$$M\vec{X}_{\text{cm}} = \int \vec{x} dm \quad \text{or} \quad \vec{X}_{\text{cm}} = \frac{1}{M} \int \vec{x} dm$$



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Momentum

Momentum is a useful idea that follows naturally from our decision to treat collections as objects. It is a way of combining the mass (which is a characteristic of the object) with the velocity of the object. We define **the momentum** to be:

$$\vec{p} = m\vec{v}$$

Thus (since the mass of an object is generally constant):

$$\vec{F} = m\vec{a} = m \frac{d\vec{v}}{dt} = \frac{d}{dt}(m\vec{v}) = \frac{d\vec{p}}{dt}$$

is another way of writing Newton's second law.

Note that there exist systems (like rocket ships, cars, etc.) where the mass is **not** constant. As the rocket rises, its thrust (the force exerted by its exhaust) can be constant, but it continually gets lighter as it burns fuel. Newton's second law (expressed as $\vec{F} = m\vec{a}$) **does** tell us what to do in this case – but only if we treat each little bit of burned and exhausted gas as a “particle”, which is a pain. On the other hand, Newton's second law expressed as $\vec{F} = \frac{d\vec{p}}{dt}$ still works fine and makes perfect sense – it simultaneously describes the loss of mass and the increase of velocity as a function of the mass correctly.



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Momentum

Clearly we can repeat our previous argument for the sum of the momenta of a collection of particles:

$$\vec{P}_{tot} = \sum_i \vec{p}_i = \sum_i m \vec{v}_i$$

so that

$$\frac{d\vec{P}_{tot}}{dt} = \sum_i \frac{d\vec{p}_i}{dt} = \sum_i \vec{F}_i = \vec{F}_{tot}$$

Differentiating our expression for the position of the center of mass above, we also get:

$$\frac{d \sum_i m \vec{x}_i}{dt} = \sum_i m \frac{d\vec{x}_i}{dt} = \sum_i \vec{p}_i = \vec{P}_{tot} = M_{tot} \vec{v}_{cm}$$



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

The Law of Conservation of Momentum

We are now in a position to state and trivially prove the Law of Conservation of Momentum.

If and only if the total external force acting on a system is zero, *then* the total momentum of a system (of particles) is a constant vector.

You are welcome to learn this in its more succinct algebraic form:

If and only if $\vec{F}_{tot} = 0$ then $\vec{P}_{tot} = \vec{P}_{initial} = \vec{P}_{final} = \text{a constant vector.}$



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Impulse



Let us imagine a typical collision: one pool ball approaches and strikes another, causing both balls to recoil from the collision in some (probably different) directions and at different speeds. Before they collide, they are widely separated and exert no force on one another.

As the surfaces of the two (hard) balls come into contact, they “suddenly” exert relatively large, relatively violent, equal and opposite forces on each other over a relatively short time, and then the force between the objects once again drops to zero as they either bounce apart or stick together and move with a common velocity.

“Relatively” here in all cases means *compared to all other forces acting on the system during the collision* in the event that those forces are not actually zero.

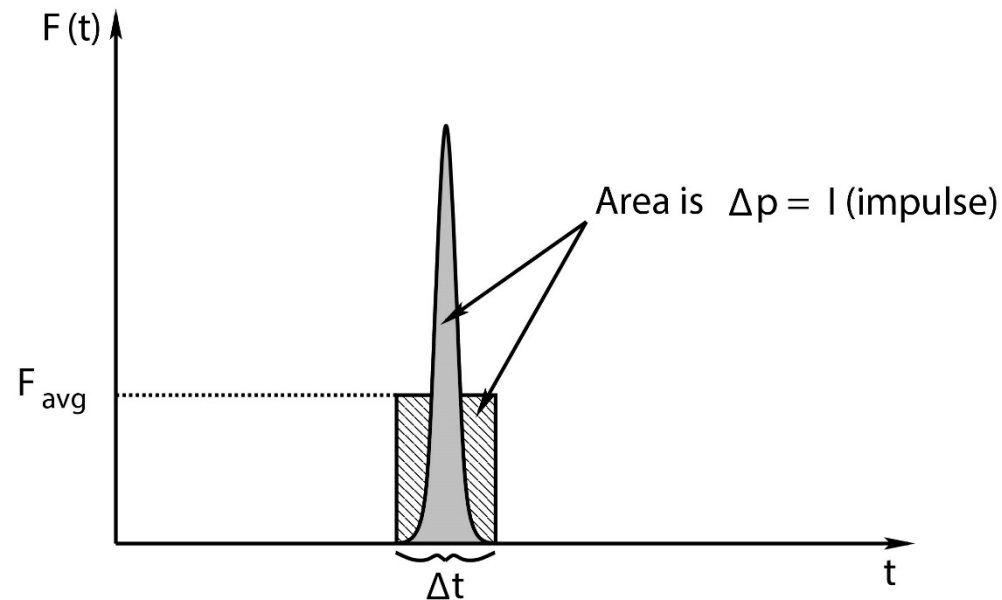


Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Impulse

Let us begin, then, by defining the average force over the (short) time Δt of any given collision, assuming that we *did* know $\vec{F} = \vec{F}_{21}(t)$, the force one object (say m_1) exerts on the other object (m_2).

The magnitude of such a force (one perhaps appropriate to the collision of pool balls) is sketched below in figure where for simplicity we assume that the force acts only along the line of contact and is hence effectively one dimensional in this direction.



The time average of this force is computed the same way the time average of any other timedependent quantity might be:



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Impulse

The time average of this force is computed the same way the time average of any other time-dependent quantity might be:

$$\vec{F}_{avg} = \frac{1}{\Delta t} \int_0^{\Delta t} \vec{F}(t) dt$$

We can evaluate the integral using Newton's Second Law expressed in terms of momentum:

$$\vec{F}(t) = \frac{d\vec{p}}{dt}$$

so that (multiplying out by dt and integrating):

$$\vec{p}_{2f} - \vec{p}_{2i} = \Delta\vec{p}_2 = \int_0^{\Delta t} \vec{F}(t) dt$$

Note that the momentum change of the first ball is equal and opposite. From Newton's Third Law, $\vec{F}_{12}(t) = -\vec{F}_{21}(t) = \vec{F}$ and:

$$\vec{p}_{1f} - \vec{p}_{1i} = \Delta\vec{p}_1 = - \int_0^{\Delta t} \vec{F}(t) dt = -\Delta\vec{p}_2$$



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Impulse

The integral of a force \vec{F} over an interval of time is called the ***impulse*** imparted by the force

$$\vec{I} = \int_{t_1}^{t_2} \vec{F}(t) dt = \int_{t_1}^{t_2} \frac{d\vec{p}}{dt}(t) dt = \int_{p_1}^{p_2} d\vec{p} = \vec{p}_2 - \vec{p}_1 = \Delta\vec{p}$$

This proves that the (vector) impulse is equal to the (vector) change in momentum over the same time interval, a result known as the ***impulse-momentum theorem***. From our point of view, the impulse is just the momentum transferred between two objects in a collision in such a way that the *total* momentum of the two is unchanged.

Returning to the average force, we see that the average force in terms of the impulse is just:

$$\vec{F}_{avg} = \frac{\vec{I}}{\Delta t} = \frac{\Delta p}{\Delta t} = \frac{\vec{p}_f - \vec{p}_i}{\Delta t}$$

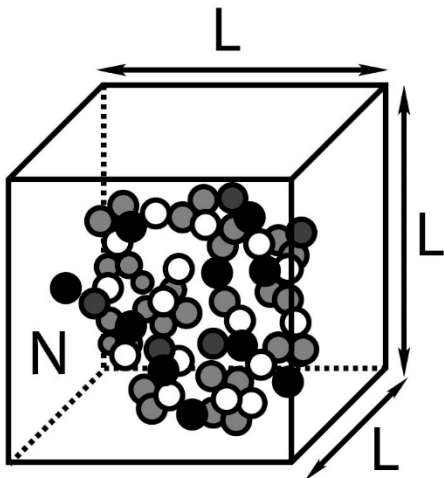


Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Impulse, Fluids, and Pressure

Another valuable use of impulse is when we have many objects colliding with something – so many that even though each collision takes only a short time Δt , there are so many collisions that they exert a nearly continuous force on the object.

This is critical to understanding the notion of pressure exerted by a fluid, because microscopically the fluid is just a lot of very small particles that are constantly colliding with a surface and thereby transferring momentum to it, so many that they exert a nearly continuous and smooth force on it that is the average force exerted per particle times the number of particles that collide.



Suppose you have a cube with sides of length L containing N molecules of a gas.



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Impulse, Fluids, and Pressure

Let's suppose that all of the molecules have a mass m and an average speed in the x direction of v_x , with (on average) one half going left and one half going right at any given time.

In order to be in equilibrium (so v_x doesn't change) the change in momentum of any molecule that hits, say, the right hand wall perpendicular to x is $\Delta p_x = 2mv_x$. This is the *impulse* transmitted to the wall per molecular collision. To find the total impulse in the time Δt , one must multiply this by one half the number of molecules in a volume $L^2 v_x \Delta t$. That is,

$$\Delta p_{tot} = \frac{1}{2} \left(\frac{N}{L^3} \right) L^2 v_x \Delta t (2mv_x)$$

Let's call the volume of the box $L^3 = V$ and the area of the wall receiving the impulse $L^2 = A$.

$$P = \frac{F_{avg}}{A} = \frac{\Delta p_{tot}}{A \Delta t} = \left(\frac{N}{V} \right) \left(\frac{1}{2} m v_x^2 \right) = \left(\frac{N}{V} \right) K_{x,avg}$$

where the average force per unit area applied to the wall is the **pressure**, which has SI units of Newtons/meter² or **Pascals**.



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Impulse, Fluids, and Pressure

If we add a result called the *equipartition theorem*:

$$K_{x,avg} = \frac{1}{2} m v_x^2 = \frac{1}{2} k_b T$$
$$\Delta p_{tot} = \frac{1}{2} \left(\frac{N}{L^3} \right) L^2 v_x \Delta t (2m v_x)$$

where k_b is Boltzmann's constant and T is the temperature in degrees absolute, one gets:

$$PV = NkT$$

which is the **Ideal Gas Law**.



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Collisions

A “collision” in physics occurs when two bodies that are more or less not interacting (because they are too far apart to interact) come “in range” of their mutual interaction force, strongly interact for a short time, and then separate so that they are once again too far apart to interact.

There are three general “types” of collision:

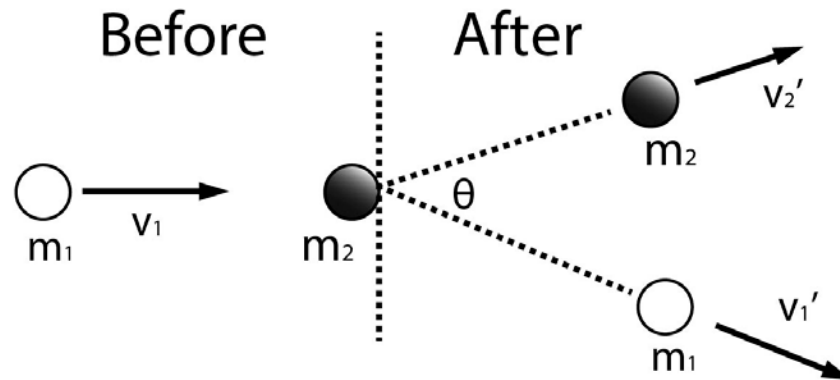
- Elastic
- Fully Inelastic
- Partially Inelastic



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Elastic collision

By definition, an *elastic collision* is one that *also* conserves *total kinetic energy* so that the total scalar kinetic energy of the colliding particles before the collision must equal the total kinetic energy after the collision. This is an additional independent equation that the solution must satisfy.



General relationships:

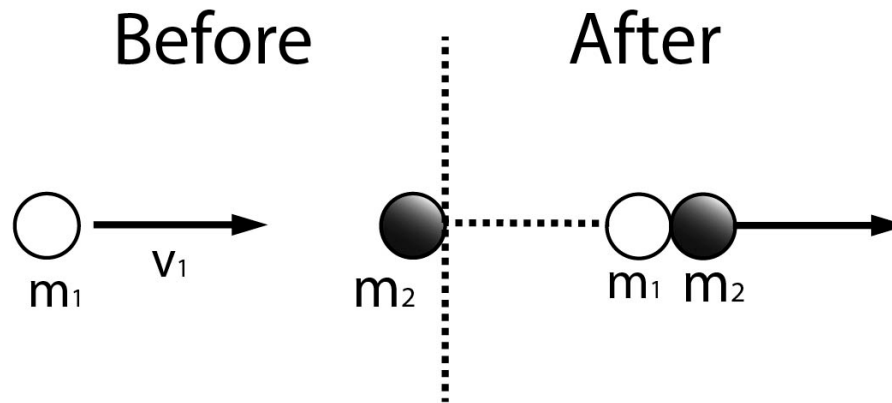
1. Conservation of momentum $\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$
2. Conservation of kinetic energy: $\frac{1}{2}m_1\vec{v}_{1i}^2 + \frac{1}{2}m_2\vec{v}_{2i}^2 = \frac{1}{2}m_1\vec{v}_{1f}^2 + \frac{1}{2}m_2\vec{v}_{2f}^2$
3. For head-on collisions: $v_1' = \frac{(m_1 - m_2)}{(m_1 + m_2)}v_1$; $v_2' = \frac{2m_1}{(m_1 + m_2)}v_1$
4. For head-on collisions the velocity of approach is equal to the velocity of separation



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Inelastic collision

A fully inelastic collision is where two particles collide and *stick* together. As always, momentum is conserved in the impact approximation, but now kinetic energy is not!



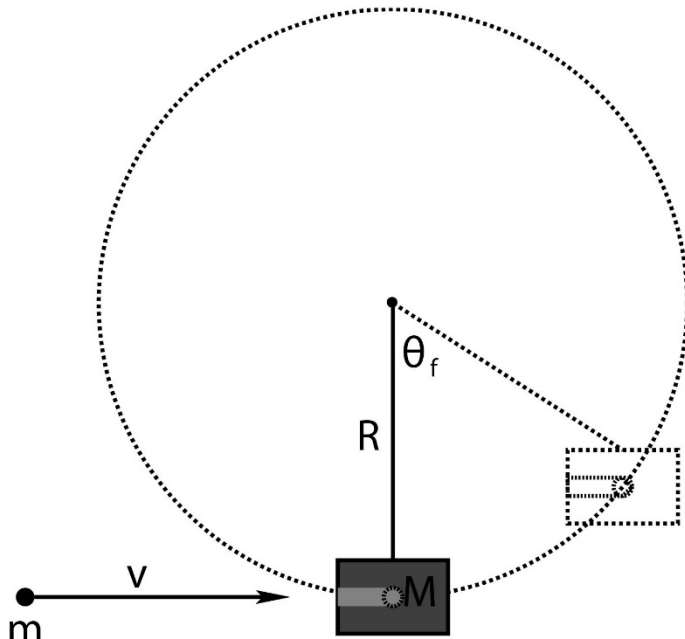
$$\vec{p}_{i,n \text{ tot}} = m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = (m_1 + m_2) \vec{v}_f = (m_1 + m_2) \vec{v}_{cm} = \vec{p}_{f,tot}$$

In other words, in a fully inelastic collision, the velocity of the outgoing combined particle is the velocity of the center of mass of the system, which we can easily compute from a knowledge of the initial momenta or velocities and masses.



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Example: Ballistic Pendulum



The “ballistic pendulum”, where a bullet strikes and sticks to/in a block, which then swings up to a maximum angle θ_f before stopping and swinging back down.

The classic ballistic pendulum question gives you the mass of the block M , the mass of the bullet m , the length of a string or rod suspending the “target” block from a free pivot, and the initial velocity of the bullet v_0 . It then asks for the maximum angle θ_f through which the pendulum swings after the bullet hits and sticks to the block (or alternatively, the maximum height H through which it swings).

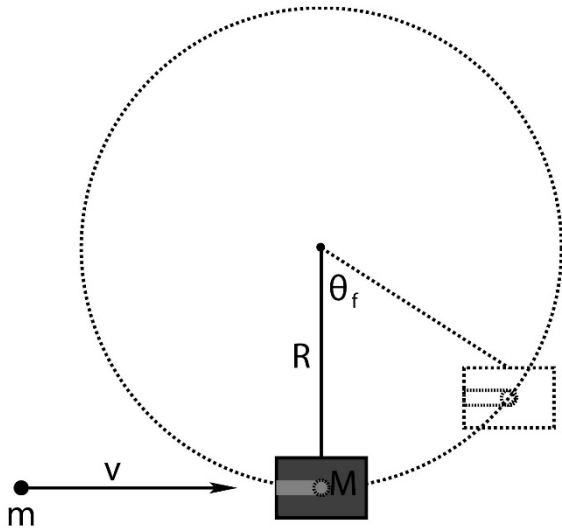
Solution:

During the collision momentum is conserved in the impact approximation, which in this case basically implies that the block has no time to swing up appreciably “during” the actual collision.



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Example: Ballistic Pendulum



Solution:

- **During the collision momentum is conserved** in the impact approximation, which in this case basically implies that the block has no time to swing up appreciably “during” the actual collision.
- **After the collision mechanical energy is conserved.** Mechanical energy is *not* conserved during the collision (see solution above of straight up inelastic collision).

Momentum conservation: $p_{m,0} = mv_0 = p_{M+m,f}$

kinetic part of mechanical energy conservation in terms of momentum:

$$E_0 = \frac{p_{B+b,f}^2}{2(M+m)} = \frac{p_{b,0}^2}{2(M+m)} = E_f = (M+m)gH = (M+m)gR(1 - \cos \theta_f)$$

Thus: $\theta_f = \cos^{-1}\left(1 - \frac{(mv_0)^2}{2(M+m)^2 gR}\right)$ which only has a solution if mv_0 is less than some maximum value.



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Torque and Rotation

Rotations in One Dimension are rotations of a solid object about a *single* axis. Since we are free to choose any arbitrary coordinate system we wish in a problem, we can without loss of generality select a coordinate system where the z -axis represents the (positive or negative) direction of rotation, so that the rotating object rotates “in” the xy plane. Rotations of a rigid body in the xy plane can then be described by a single angle θ , measured by convention in the counterclockwise direction from the positive x -axis.

Time-dependent Rotations can thus be described by:

- a) The *angular position* as a function of time, $\theta(t)$.
- b) The *angular velocity* as a function of time,

$$w(t) = \frac{d\theta}{dt}$$

- c) The *angular acceleration* as a function of time,

$$\alpha(t) = \frac{dw}{dt} = \frac{d^2\theta}{dt^2}$$



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Torque and Rotation

- Forces applied to a rigid object perpendicular to a line drawn from an *axis of rotation* exert a **torque** on the object. The torque is given by:

$$\tau = rF \sin(\varphi) = rF_{\perp} = r_{\perp}F$$

- The torque (as we shall see) is a *vector* quantity and by convention its direction is *perpendicular* to the plane containing \vec{r} and \vec{F} in the direction given by **the right hand rule**. Although we won't really work with this until next week, the “proper” definition of the torque is:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

- Newton's Second Law for Rotation** in one dimension is:

$$\tau = I\alpha$$

where I is the moment of inertia of the rigid body being rotated by the torque about a given/specified axis of rotation. The direction of this (one dimensional) rotation is the righthanded direction of the axis – the direction your right handed thumb points if you grasp the axis with your fingers curling around the axis in the direction of the rotation or torque.



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Torque and Rotation

- The *moment of inertia of a point particle* of mass m located a (fixed) distance r from some axis of rotation is:

$$I = mr^2$$

- The moment of inertia of a rigid collection of point particles is:

$$I = \sum_i m_i r_i^2$$

- The moment of inertia of a continuous solid rigid object is:

$$I = \int r^2 dm$$

- The rotational kinetic energy of a rigid body (total kinetic energy of all of the chunks of mass that make it up) is:

$$K_{rot} = \frac{1}{2} I \omega^2$$



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Conditions for Static Equilibrium

An object at rest remains at rest unless acted on by a net external force.

Previously we showed that *Newton's Second Law* also applies to *systems of particles*, with the replacement of the position of the particle by the position of the *center of mass* of the system and the force with the total external force acting on the entire system.

We also learned that the *force equilibrium* of particles acted on by conservative force occurred at the points where the potential energy was maximum or minimum or neutral (flat), where we named maxima “unstable equilibrium points”, minima “stable equilibrium points” and flat regions “neutral equilibria”.

However, we learned enough to now be able to see that force equilibrium *alone* is ***not sufficient*** to cause an extended object or collection of particles to be in equilibrium. We can easily arrange situations where *two* forces act on an object in opposite directions (so there is no net force) but along lines such that together they exert a nonzero *torque* on the object and hence cause it to angularly accelerate and gain kinetic energy without bound, hardly a condition one would call “equilibrium”.



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Conditions for Static Equilibrium

The Newton's Second Law for Rotation is sufficient to imply Newton's First Law for Rotation:

If, in an inertial reference frame, a rigid object is initially at rotational rest (not rotating), it will remain at rotational rest unless acted upon by a net external torque.

That is, $\vec{\tau} = I\vec{\alpha} = 0$ implies $\vec{\omega} = 0$ and constant. We will call the condition where $\vec{\tau} = 0$ and a rigid object is not rotating *torque equilibrium*.

Therefore we now *define* the conditions for the *static equilibrium of a rigid body* to be:

A rigid object is in static equilibrium when both the vector torque and the vector force acting on it are zero.

That is:

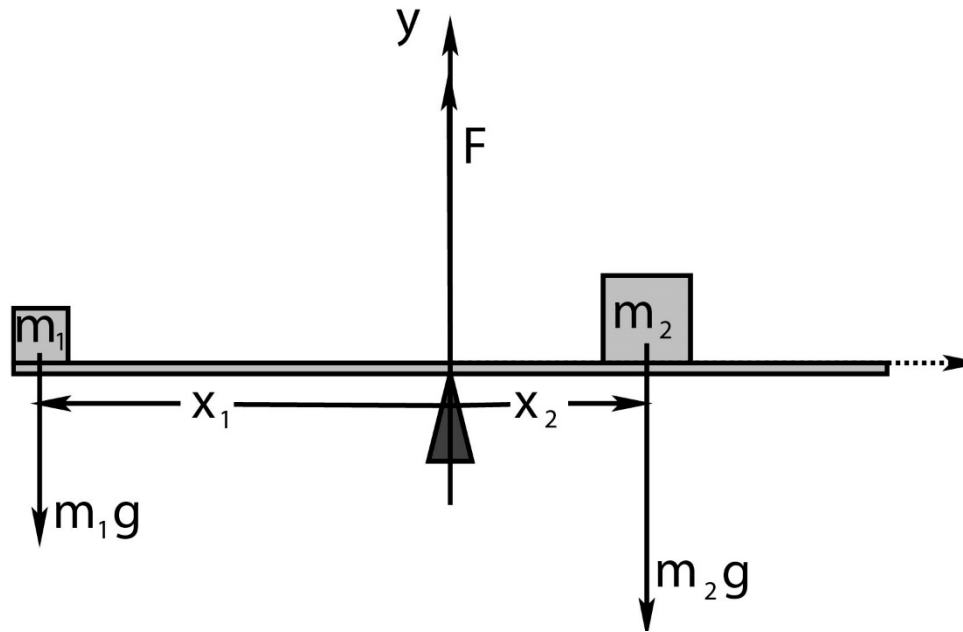
If $\vec{F}_{tot} = 0$ and $\vec{\tau}_{tot} = 0$, then an object initially at translational and rotational rest will remain at rest and neither accelerate nor rotate.



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Balancing a See-Saw

You are given m_1 , x_1 , and x_2 and are asked to find m_2 and F such that the see-saw is in *static equilibrium*.



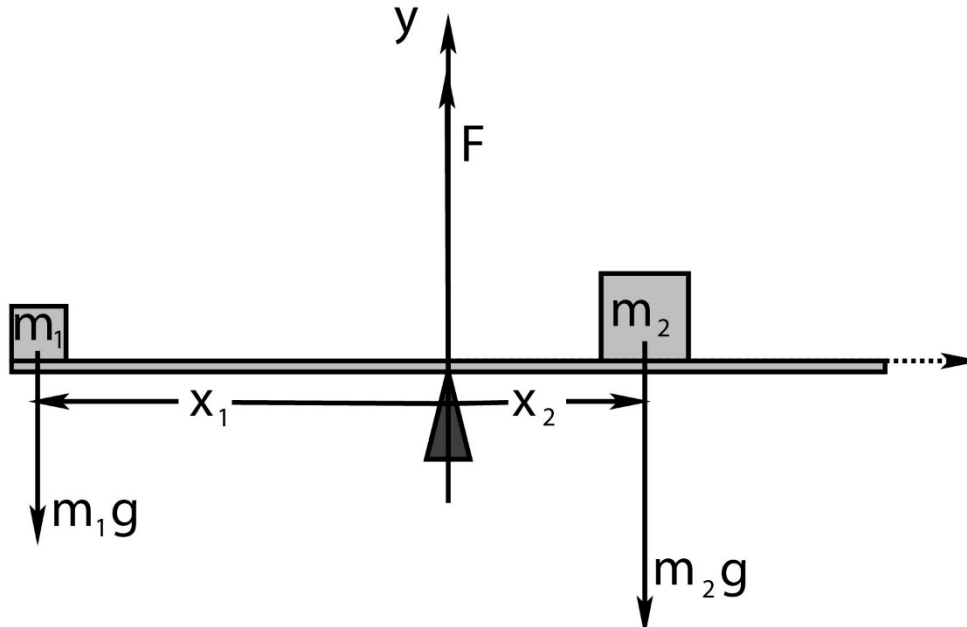
One typical problem in statics is balancing weights on a see-saw type arrangement – a uniform plank supported by a fulcrum in the middle. This particular problem is really only one dimensional as far as force is concerned, as there is no force acting in the x -direction or z -direction.



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Balancing a See-Saw

Let's imagine that in this particular problem, the mass m_1 and the distances x_1 and x_2 are given, and we need to find m_2 and F .



We have two choices to make – where we select the pivot and which direction (in or out of the page) we are going to define to be “positive”. A perfectly reasonable choice is to select the pivot at the fulcrum of the see-saw where the unknown force F is exerted, and to select the $+z$ -axis as positive rotation.

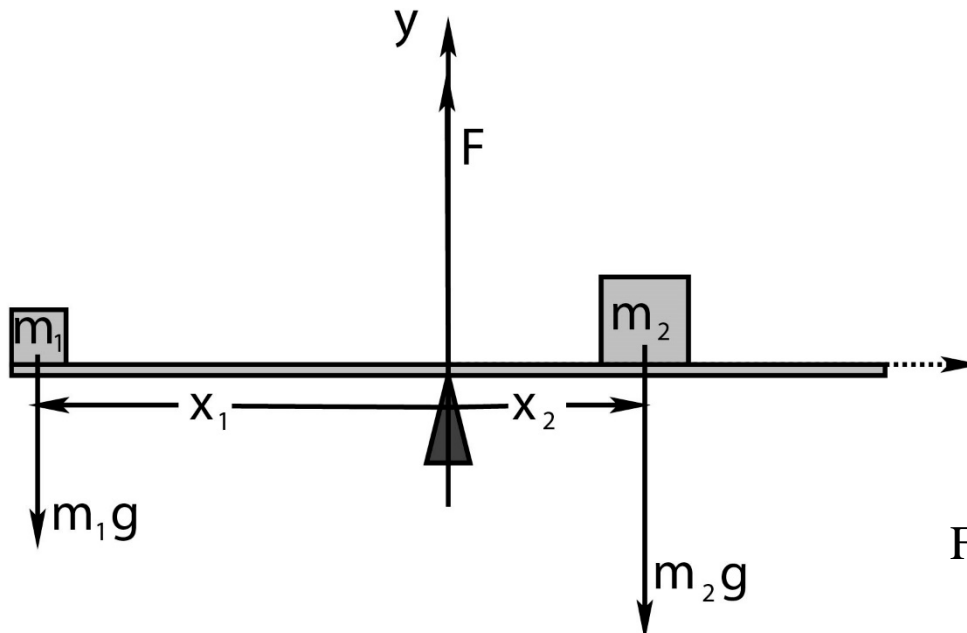
$$\sum F_y = F - m_1g - m_2g = 0$$

$$\sum \tau_z = x_1m_1g - x_2m_2g = 0$$



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Balancing a See-Saw



$$\sum F_y = F - m_1g - m_2g = 0$$

$$\sum \tau_z = x_1 m_1 g - x_2 m_2 g = 0$$

$$m_2 = \frac{m_1 g x_1}{g x_2} = \left(\frac{x_1}{x_2} \right) m_1$$

From the first equation and the solution for m_2 :

$$F = m_1 g + m_2 g = m_1 g \left(1 + \left(\frac{x_1}{x_2} \right) \right) = m_1 g \left(\frac{x_1 + x_2}{x_2} \right)$$

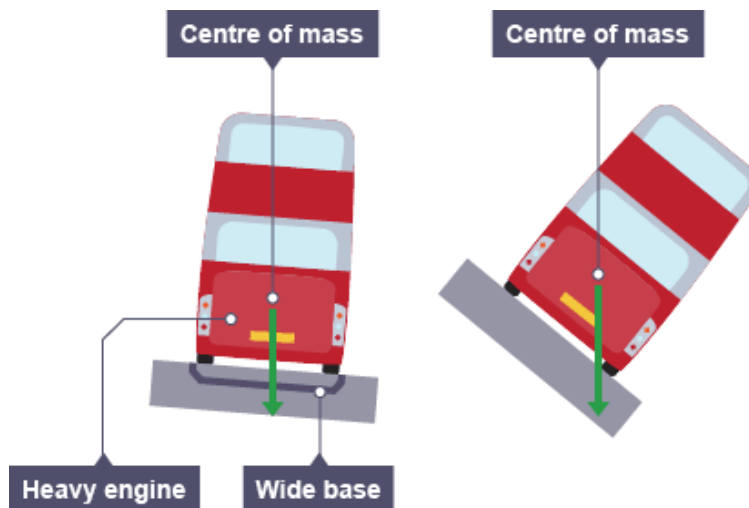


Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Tipping

Another important application of the ideas of static equilibrium is to ***tipping problems***. A tipping problem is one where one uses the ideas of static equilibrium to identify the particular angle or force combination that will *marginally* cause some object to tip over.

The idea of tipping is simple enough. An object placed on a flat surface is typically stable as long as the center of gravity is vertically ***inside the edges*** that are in contact with the surface, so that the torque created by the gravitational force around this limiting pivot is opposed by the torque exerted by the (variable) normal force.

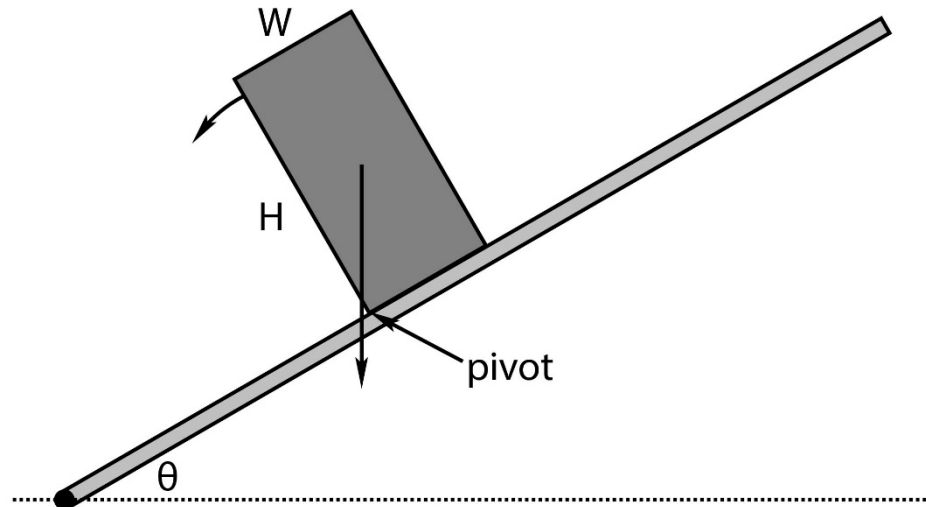




Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Tipping Versus Slipping

A rectangular block either tips first or slips (slides down the incline) first as the incline is gradually increased. Which one happens first? The figure is show with the block just past the *tipping angle*.



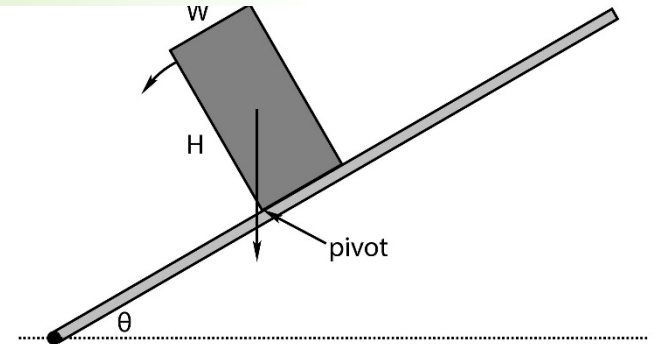
At some angle we know that the block will start to slide. This will occur because the normal force is decreasing with the angle (and hence, so is the maximum force static friction can exert) and at the same time, the component of the weight of the object that points *down* the incline is increasing. Eventually the latter will exceed the former and the block will slide. However, at some angle the block will *also* tip over. We know that this will happen because the normal force can only prevent the block from rotating *clockwise* (as drawn) around the pivot consisting of the lower left corner of the block.



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

Tipping Versus Slipping

The *tipping point*, or *tipping angle* is thus the angle where the *center of gravity* is directly *over the pivot* that the object will “tip” around as it falls over.



Let's find the slipping angle θ_s . Let “down” mean “down the incline”. Then:

$$\sum F_{\text{down}} = mg \sin(\theta) - F_s = 0$$

$$\sum F_{\perp} = N - mg \cos(\theta) = 0$$

From the latter, as usual: $N = mg \cos(\theta)$ and $F_s \leq F_s^{\text{max}} = \mu_s N$

When $mg \sin(\theta_s) = F_s^{\text{max}} = \mu_s N \cos(\theta_s)$

The force of gravity down the incline precisely balances the force of static friction. We can solve for the angle where this occurs: $\theta_s = \tan^{-1}(\mu_s)$

This happens when the center of mass passes *directly over the pivot*.



Lecture 4. Systems of Particles, Momentum and Collisions. Statics

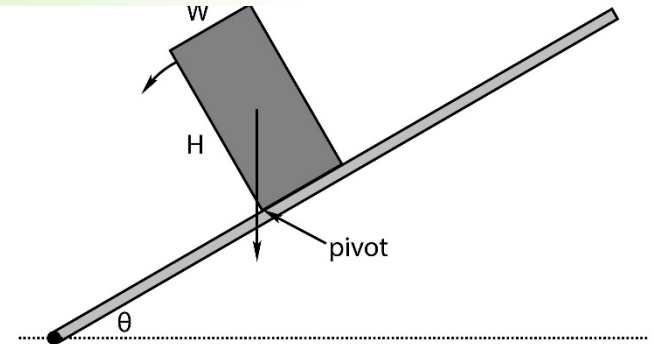
Tipping Versus Slipping

From inspection of the figure (which is drawn *very close* to the tipping point) it should be clear that the tipping angle θ_t is given by:

$$\theta_t = \tan^{-1} \left(\frac{W}{H} \right)$$

So, which one wins? The smaller of the two, θ_s or θ_t , of course – that's the one that happens first as the plank is raised. Indeed, since both are inverse tangents, the smaller of: μ_s , W/H

determines whether the system slips first or tips first, no need to actually *evaluate* any tangents or inverse tangents!

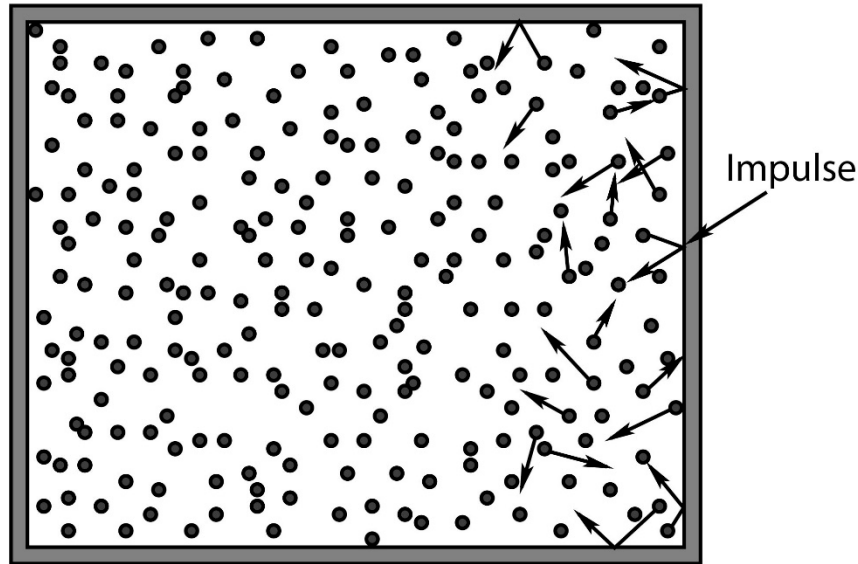




Lecture 5. Fluids

General Fluid Properties

Fluids are the generic name given to two states of matter, liquids and gases characterized by a lack of long range order and a high degree of mobility at the molecular scale.

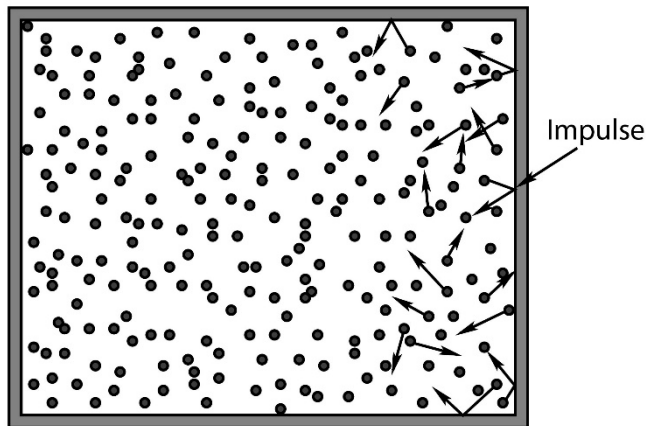


A large number of atoms or molecules are confined within in a “box”, where they bounce around off of each other and the walls. They exert a *force* on the walls equal and opposite the force the walls exert on them as the collisions more or less elastically reverse the particles’ momenta perpendicular to the walls.



Lecture 5. Fluids

General Fluid Properties



Many particles all of mass m are *constantly moving* in random, constantly changing directions (as the particles collide with each other and the walls) with an average kinetic energy related to the temperature of the fluid. Some of the particles (which might be atoms such as helium or neon or molecules such as H_2 or O_2) happen to be close to the walls of the container and moving in the right direction to bounce (elastically) off of those walls.

When they do, their *momentum* perpendicular to those walls is *reversed*. Since *many, many* of these collisions occur each second, there is a nearly continuous momentum transfer between the walls and the gas and the gas and the walls. This transfer, per unit time, becomes the *average force* exerted by the walls on the gas and the gas on the walls

Eventually, we will transform this simple picture into the ***Kinetic Theory of Gases*** and use it to derive the venerable ***Ideal Gas Law***

$$PV = Nk_bT$$



Lecture 5. Fluids

Pressure

To describe the forces that confine and act on the fluids in terms of **pressure**, defined to be the **force per unit area** with which a fluid pushes on a confining wall or the confining wall pushes on the fluid:

$$P = \frac{F}{A}$$

Pressure gets its own SI units, which clearly must be Newtons per square meter. We give these units their own name, **Pascals**:

$$1 \text{ Pascal} = \frac{\text{Newton}}{\text{meter}^2}$$

A Pascal is a tiny unit of pressure – a Newton isn't very big, recall (one kilogram weighs roughly ten Newtons) so a Pascal is the weight of a quarter pound spread out over a square meter.

A more convenient measure of pressure in our everyday world is a form of the unit called a **bar**:

$$1 \text{ bar} = 10^5 \text{ Pa} = 100 \text{ kPa}$$



Lecture 5. Fluids

Pressure

The average air pressure at sea level is very nearly 1 bar.

The symbol *atm* stands for *one standard atmosphere*. The connection between atmospheres, bars, and pascals is:

$$1 \text{ standard atmosphere} = 101.325 \text{ kPa} = 1013.25 \text{ mbar}$$

The extra significant digits therefore refer only to a fairly arbitrary value (in pascals) historically related to the original definition of a standard atmosphere in terms of “millimeters of mercury” or *torr* :

$$1 \text{ standard atmosphere} = 760.00 \text{ mmHg} = 760.00 \text{ torr}$$

In this class we will use the simple rule $1 \text{ bar} \approx 1 \text{ atm}$

Note well: in the field of medicine blood pressures are given in mm of mercury (or torr) by long standing tradition (largely because for at least a century blood pressure was measured with a mercury-based sphygmomanometer). These can be converted into atmospheres by dividing by 760, remembering that one is measuring the difference between these pressures and the standard atmosphere (so the actual blood pressure is always greater than one atmosphere).



Lecture 5. Fluids

Density

Even a very tiny volume of fluid has many, many atoms or molecules in it.

We can work to create a *vacuum* – a volume that has relatively *few* molecules in it per unit volume, but it is almost impossible to make that number zero – even the hard vacuum of outer space has on average one molecule per cubic meter or thereabouts. We live at the bottom of a gravity well that confines our *atmosphere* – the air that we breathe – so that it forms a relatively thick soup that we move through and breathe with order of *Avogadro's Number* (6×10^{23}) molecules per *liter* – hundreds of billions of billions per cubic centimeter.

At this point we cannot possibly track the motion and interactions of all of the individual molecules, so we *coarse grain* and *average*.

The properties of *oxygen* molecules and *helium* molecules might well be very different, so the molecular count alone may not be the most useful quantity. Since we are interested in how forces might act on these small volumes, we need to know their mass, and thus we define the *density* of a fluid to be:

$$\rho = \frac{dm}{dV}$$



Lecture 5. Fluids

Compressibility

A major difference between fluids and solids, and liquids and gases within the fluids, is the ***compressibility*** of these materials. Compressibility describes how a material responds to changes in *pressure*.

This can be expressed as a simple linear relationship:

$$\Delta P = -B \frac{\Delta V}{V}$$

Pressure up, volume down and vice versa. The constant of proportionality B is called the ***bulk modulus*** of the material.

Note well that we haven't really specified *yet* whether the “material” is solid, liquid or gas. All three of them have densities, all three of them have bulk moduli. Where they differ is in the *qualitative* properties of their compressibility.



Lecture 5. Fluids

Compressibility

- **Solids** are typically *relatively* incompressible (large B), although there are certainly exceptions. They have long range order – all of the molecules are packed and tightly bonded together in structures and there is usually very little free volume.
- **Liquids** are also relatively incompressible (large B). They differ from solids in that they lack long range order. All of the molecules are constantly moving around and any small “structures” that appear due to local interaction are short-lived. The molecules of a liquid are close enough together that there is often significant physical and chemical interaction, giving rise to surface tension and wetting properties – especially in water, which is an amazing fluid!
- **Gases** are in contrast quite *compressible* (small B). One can usually squeeze gases smoothly into smaller and smaller volumes, until they reach the point where the molecules are basically all touching and the gas converts to a liquid! Gases per se (especially hot gases) usually remain “weakly interacting” right up to where they become a liquid, although the correct (non-ideal) equation of state for a real gas often displays features that are the results of moderate interaction, depending on the pressure and temperature.



Lecture 5. Fluids

Compressibility

Water is, as noted, a remarkable liquid. H_2O is a polar molecules with a permanent dipole moment, so water molecules are very strongly interacting, both with each other and with other materials. It organizes itself quickly into a state of relative order that is very *incompressible*.



The bulk modulus of water is 2.2×10^9 Pa, which means that even deep in the ocean where pressures can be measured in the tens of millions of Pascals (or hundreds of atmospheres) the density of water only varies by a few percent from that on the surface. Its density varies much more rapidly with *temperature* than with pressure.

We will idealize water by considering it to be *perfectly incompressible* in this course, which is close enough to true for nearly any mundane application of hydraulics that you are most unlikely to ever observe an exception that matters.



Lecture 5. Fluids

Viscosity and fluid flow

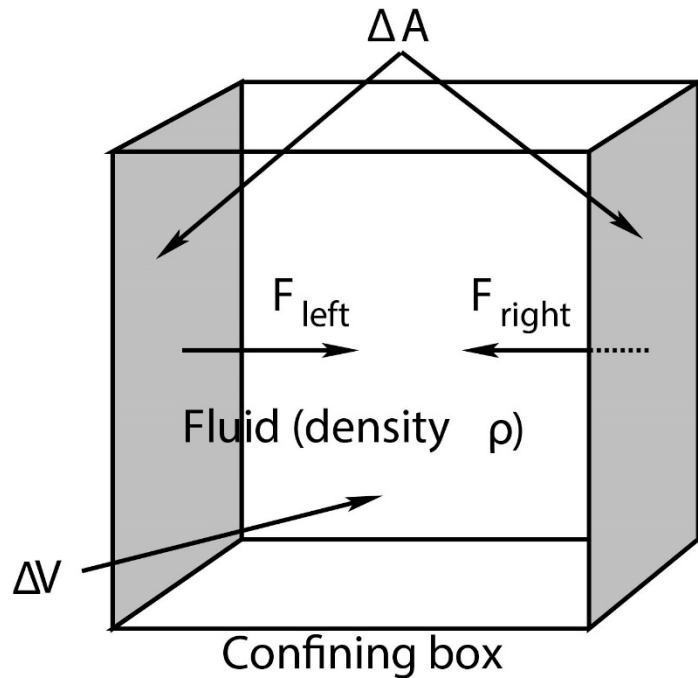
Fluids, whether liquid or gas, have some internal “stickiness” that resists the relative motion of one part of the fluid compared to another, a kind of internal “friction” that tries to equilibrate an entire body of fluid to move together. They also interact with the walls of any container in which they are confined.

The **viscosity** of a fluid (symbol μ) is a measure of this internal friction or stickiness. Thin fluids have a low viscosity and flow easily with minimum resistance; thick sticky fluids have a high viscosity and resist flow.

Fluid, when flowing through (say) a cylindrical pipe tends to organize itself in one of two very different ways – a state of *laminar flow* where the fluid at the very edge of the flowing volume is at rest where it is in contact with the pipe and the speed concentrically and symmetrically increases to a maximum in the center of the pipe, and *turbulent flow* where the fluid tumbles and rolls and forms eddies as it flows through the pipe. Turbulence and flow and viscosity are properties that will be discussed in more detail below.

Lecture 5. Fluids

Static Fluids. Pressure and Confinement of Static Fluids

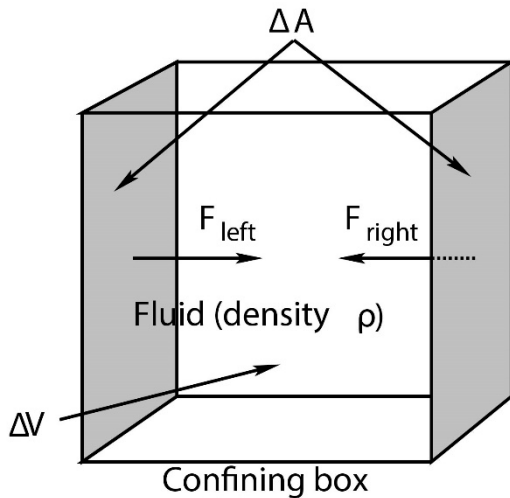


In figure we see a box of a fluid that is confined within the box by the rigid walls of the box.

We will imagine that this particular box is in “free space” far from any gravitational attractor and is therefore at rest with no external forces acting on it. We know from our intuition based on things like cups of coffee that no matter how this fluid is initially stirred up and moving within the container, after a very long time the fluid will damp down any initial motion by interacting with the walls of the container and arrive at *static equilibrium*.

Lecture 5. Fluids

Static Fluids. Pressure and Confinement of Static Fluids



A fluid in static equilibrium has the property that every single tiny chunk of volume in the fluid has to *independently* be in force *equilibrium* – the total force acting on the differential volume chunk must be *zero*.

In addition the net torques acting on all of these differential subvolumes must be zero, and the fluid must be at rest, neither translating nor rotating.

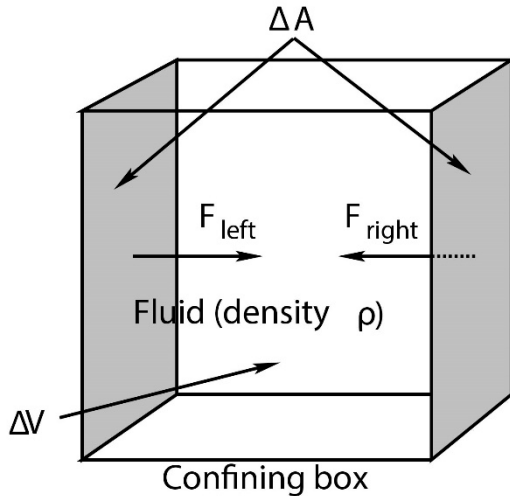
Fluid rotation is more complex than the rotation of a static object because a fluid can be *internally* rotating even if all of the fluid in the outermost layer is in contact with a contain and is *stationary*. It can also be *turbulent* – there can be lots of internal eddies and swirls of motion, including some that can exist at very small length scales and persist for fair amounts of time.

We will idealize all of this – when we discuss static properties of fluids we will assume that all of this sort of internal motion has disappeared.



Lecture 5. Fluids

Static Fluids. Pressure and Confinement of Static Fluids



We can now make a few very simple observations about the forces exerted by the walls of the container on the fluid within. First of all the mass of the fluid in the box above is clearly:

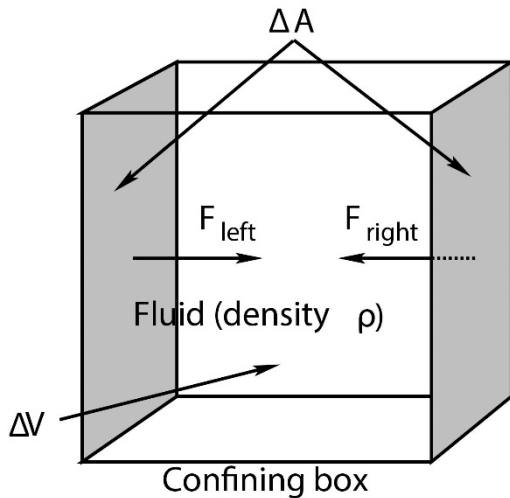
$$\Delta M = \rho \Delta V$$

We drew a symmetric box to make it easy to see that the magnitudes of the forces exerted by opposing walls are equal $F_{\text{left}} = F_{\text{right}}$ (for example). Similarly the forces exerted by the top and bottom surfaces, and the front and back surfaces, must cancel.

Suppose (as shown) the cross-sectional area of the left and right walls are ΔA originally. Consider now what we expect if we *double* the size of the box and at the same time add enough additional fluid for the fluid density to remain the same, making the side walls have the area $2 \Delta A$. With twice the area (and twice the volume and twice as much fluid), we have twice as many molecular collisions per unit time on the doubled wall areas (with the same average impulse per collision). The average force exerted by the doubled wall areas therefore *also doubles*.

Lecture 5. Fluids

Static Fluids. Pressure and Confinement of Static Fluids



From this simple argument we can conclude that the average force exerted by any wall is proportional to the area of the wall. This force is therefore most naturally expressible in terms of pressure:

$$F_{\text{left}} = P_{\text{left}} \Delta A = P_{\text{right}} \Delta A = F_{\text{right}}$$

which implies that the pressure at the left and right confining walls is the same:

$$P_{\text{left}} = P_{\text{right}} = P$$

An important property of fluids is that *one part of a fluid can move independent of another* so the fluid in at least *some* layer with a finite thickness near the wall would therefore experience a *net* force and would *accelerate*. But this violates our assumption of static equilibrium, so a fluid in *static equilibrium* exerts no tangential force on the walls of a confining container and vice versa.

We therefore conclude that the direction of the force exerted by a confining surface with an area ΔA on the fluid that is in contact with it is: $\vec{F} = P \Delta A \hat{n}$. Where \hat{n} is an inward-directed unit vector *perpendicular* to (normal to) the surface.



Lecture 5. Fluids

Pressure and Confinement of Static Fluids in Gravity

The principle change brought about by setting our box of fluid down on the ground in a gravitational field is that an additional external force comes into play: The weight of the fluid. A static fluid, confined in some way in a gravitational field, must ***support the weight of its many component parts*** internally, and of course the box itself must support the weight of the entire mass ΔM of the fluid.

As hopefully you can see if you carefully read the previous section. The only force available to provide the necessary internal support or confinement force is the ***variation of pressure within the fluid***. We would like to know how the pressure varies as we move up or down in a static fluid so that it supports its own weight.

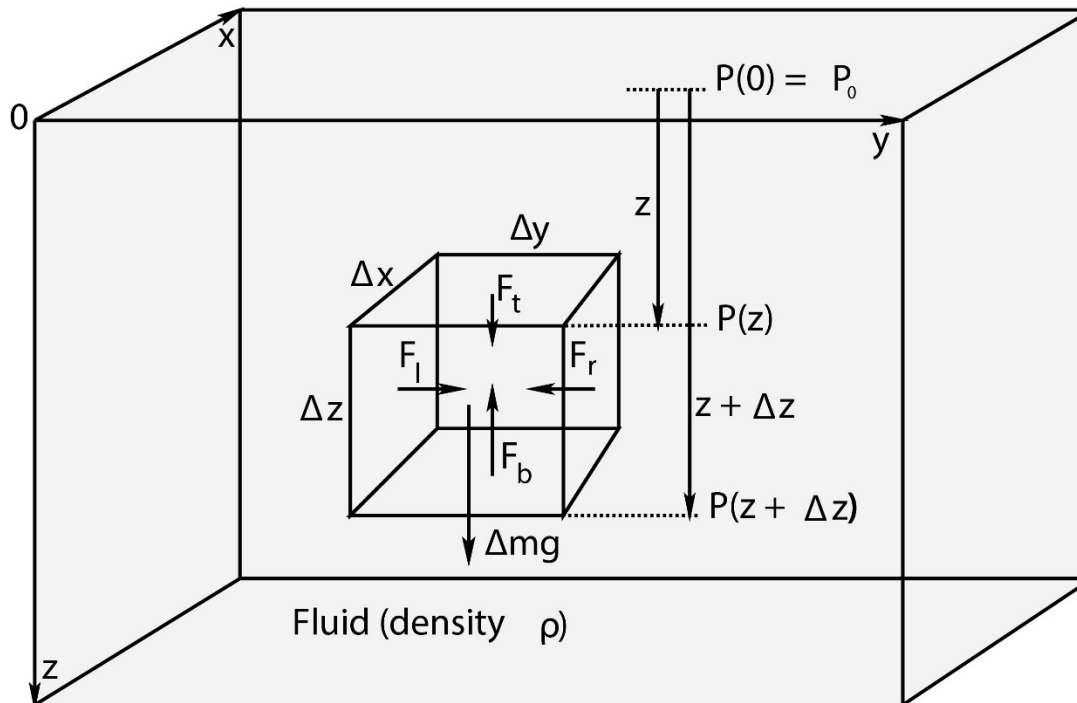
If we consider a tiny (eventually differentially small) chunk of fluid in force equilibrium, gravity will pull it down and the only thing that can push it up is a pressure difference between the top and the bottom of the chunk.



Lecture 5. Fluids

Pressure and Confinement of Static Fluids in Gravity

A fluid in static equilibrium confined to a sealed rectilinear box in a near-Earth gravitational field \vec{g} . Note well the small chunk of fluid with dimensions Δx , Δy , Δz in the middle of the fluid. Also note that the coordinate system selected has z increasing from the top of the box down, so that z can be thought of as the depth of the fluid.





Lecture 5. Fluids

Pressure and Confinement of Static Fluids in Gravity

In figure a (portion of) a fluid confined to a box is illustrated. The box could be a completely sealed one with rigid walls on all sides, or it could be something like a cup or bucket that is open on the top but where the fluid is still confined there by e.g. atmospheric pressure.

Let us consider a small (eventually infinitesimal) chunk of fluid somewhere in the *middle* of the container. As shown, it has physical dimensions Δx , Δy , Δz ; its upper surface is a distance z below the origin (where z increases down and hence can represent “depth”) and its lower surface is at depth $z + \Delta z$. The areas of the top and bottom surfaces of this small chunk are e.g. $\Delta A_{tb} = \Delta x \Delta y$, the areas of the sides are $\Delta x \Delta z$ and $\Delta y \Delta z$ respectively, and the volume of this small chunk is $\Delta V = \Delta x \Delta y \Delta z$.

This small chunk is itself in static equilibrium – therefore the forces between any pair of its horizontal sides (in the x or y direction) must cancel. As before (for the box in space) $F_l = F_r$ in magnitude (and opposite in their y -direction) and similarly for the force on the front and back faces in the x -direction, which will always be true if the pressure does not vary horizontally with variations in x or y . In the z -direction, however, force equilibrium requires that:

$$F_t + \Delta m g - F_b = 0$$



Lecture 5. Fluids

Pressure and Confinement of Static Fluids in Gravity

The only possible source of F_t and F_b are the pressure in the fluid itself which will vary with the depth z : $F_t = P(z)\Delta A_{tb}$ and $F_b = P(z + \Delta z)\Delta A_{tb}$. Also, the mass of fluid in the (small) box is $\Delta m = \rho\Delta V$ (using our ritual incantation “the mass of the chunks is...”). We can thus write:

$$P(z)\Delta x\Delta y + \rho(\Delta x\Delta y\Delta z)g - P(z + \Delta z)\Delta x\Delta y = 0$$

$$\frac{\Delta P}{\Delta z} = \frac{P(z + \Delta z) - P(z)}{\Delta z} = \rho g$$

Finally, we take the limit $\Delta z \rightarrow 0$ and identify the *definition of the derivative* to get:

$$\frac{dP}{dz} = \rho g$$

Identical arguments but without any horizontal external force followed by $\Delta x \rightarrow 0$ and $\Delta y \rightarrow 0$ lead to:

$$\frac{dP}{dx} = \frac{dP}{dy} = 0$$

as well – P does not vary with x or y as already noted



Lecture 5. Fluids

Pressure and Confinement of Static Fluids in Gravity

$$\frac{dP}{dz} = \rho g$$

In order to find $P(z)$ from this differential expression (which applies, recall, to *any* confined fluid in static equilibrium in a gravitational field) we have to *integrate* it. This integral is very simple if the fluid is incompressible because in that case **ρ is a constant**. The integral isn't *that* difficult if ρ is *not* a constant as implied by the equation we wrote above for the bulk compressibility.

We will therefore first do incompressible fluids, then compressible ones.



Lecture 5. Fluids

Variation of Pressure in Incompressible Fluids

In the case of incompressible fluids, ρ is a constant and does not vary with pressure and/or depth. Therefore we can easily multiply $dP/dz = \rho g$ above by dz on both sides and integrate to find P :

$$dP = \rho g \, dz$$

$$\int dP = \int \rho g \, dz$$

$$P(z) = \rho g z + P_0$$

where P_0 is the *constant of integration* for both integrals, and practically speaking is the pressure in the fluid at zero depth (wherever that might be in the coordinate system chosen).

Lecture 5. Fluids

Barometers

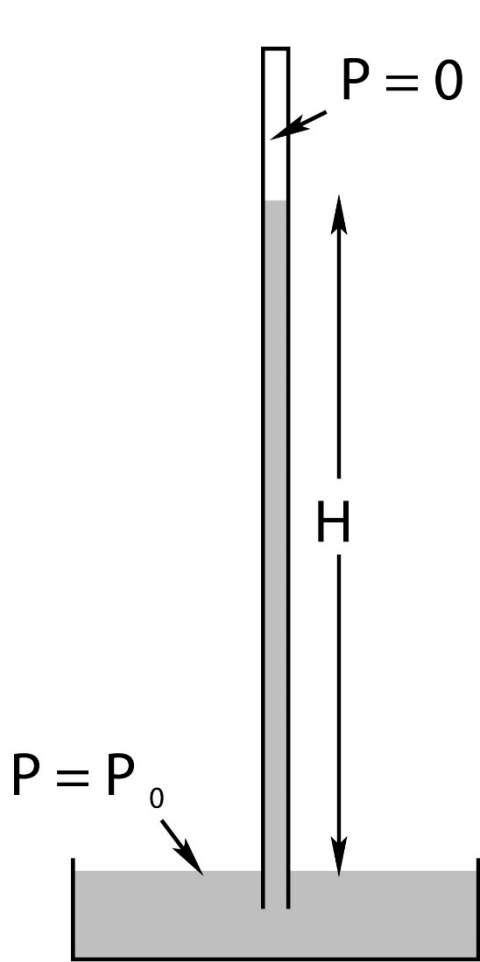


Mercury barometers were originally invented by *Evangelista Torricelli* a natural philosopher who acted as Galileo's secretary for the last three months of Galileo's life under house arrest.

Torricelli demonstrated that a shorter glass tube filled with mercury, when inverted into a dish of mercury, would fall back into a column with a height of roughly 0.76 meters with a vacuum on top, and soon thereafter discovered that the height of the column fluctuated with the pressure of the outside air pressing down on the mercury in the dish, correctly concluding that water would behave exactly the same way.

Lecture 5. Fluids

Barometers



A simple mercury barometer is shown in figure. It consists of a tube that is completely filled with mercury. Mercury has a specific gravity of 13.534 at a typical room temperature, hence a density of 13534 kg/m^3 . The filled tube is then inverted into a small reservoir of mercury. The mercury falls (pulled down by gravity) out of the tube, leaving behind a vacuum at the top. We can easily compute the expected height of the mercury column if P_0 is the pressure on the exposed surface of the mercury in the reservoir. In that case:

$$P = P_0 + \rho g z$$

as usual for an incompressible fluid. Applying this formula to both the top and the bottom, $P(0) = P_0$ and

$$P(H) = P_0 - \rho g H$$

$$P_0 = \rho g H$$

and one can easily convert the measured height H of mercury above the top surface of mercury in the reservoir into P_0 , the air pressure on the top of the reservoir.



Lecture 5. Fluids

Barometers

At one standard atmosphere, we can easily determine what a mercury barometer at room temperature will read (the height H of its column of mercury above the level of mercury in the reservoir):

$$P_0 = 13534 \frac{\text{kg}}{\text{m}^3} \times 9.80665 \frac{\text{m}}{\text{sec}^2} \times H = 101325 \text{ Pa}$$

Dividing we find the value of H expected at one standard atmosphere:

$$H_{\text{atm}} = 0.76000 = 760.00 \text{ millimeters}$$

$$P(H) = P_0 - \rho g H$$

$$P_0 = \rho g H$$

and one can easily convert the measured height H of mercury above the top surface of mercury in the reservoir into P_0 , the air pressure on the top of the reservoir.



Lecture 5. Fluids

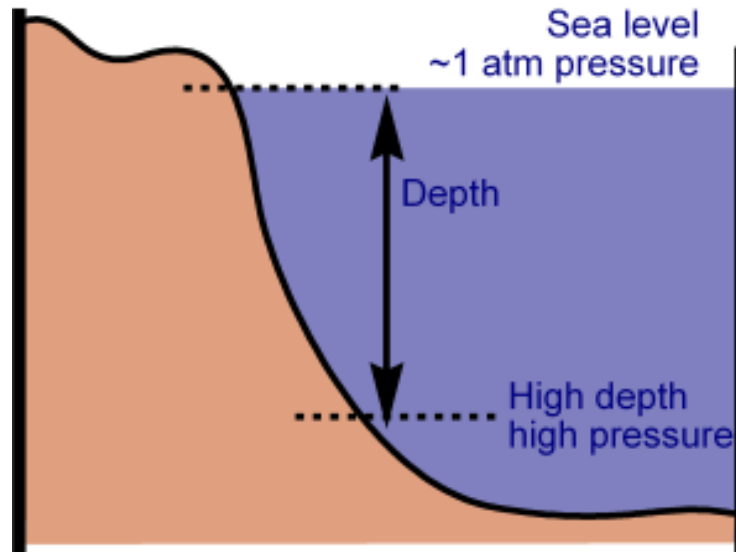
Variation of Oceanic Pressure with Depth

The pressure on the surface of the ocean is, approximately, by definition, one atmosphere. Water is a highly incompressible fluid with $\rho_w = 1000$ kilograms per cubic meter. $g \approx 10$ meters/second². Thus:

$$P(z) = P_0 + \rho_w g z = (10^5 + 10^4 z) \text{ Pa}$$

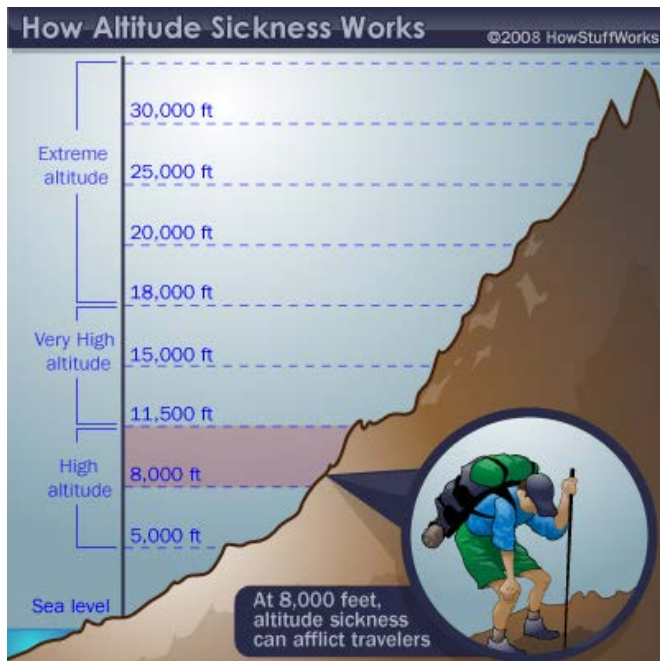
$$\text{or } P(z) = (1.0 + 0.1z) \text{ bar} = (1000 + 100z) \text{ mbar}$$

Every ten meters of depth (either way) increases water pressure by (approximately) one atmosphere!



Lecture 5. Fluids

Variation of Atmospheric Pressure with Height



<http://adventure.howstuffworks.com/outdoor-activities/climbing/altitude-sickness1.htm>

Using z to describe depth is moderately inconvenient, so let us define the height h above sea level to be $-z$. In that case P_0 is 1 Atmosphere. The molar mass of dry air is $M = 0.029$ kilograms per mole. $R = 8.31$ Joules/(mole- K°). Hence a bit of multiplication at $T = 300^\circ$:

$$\frac{M g}{RT} = \frac{0.029 \times 10}{8.31 \times 300} = 1.12 \times 10^{-4} \text{ meters}^{-1}$$

$$P(h) = 10^5 \exp(-0.00012 h) \text{ Pa} \\ = 1000 \exp(-0.00012 h) \text{ mbar}$$

This equation predicts that air pressure should drop to $1/e$ of its sea-level value of 1000 mbar at a height of around 8000 meters, the height of the so-called *death zone*. We can compare the actual (average) pressure at 8000 meters, 356 mbar, to $1000 \times e^{-1} = 368$ mbar.



Lecture 5. Fluids

Pascal's Principle and Hydraulics

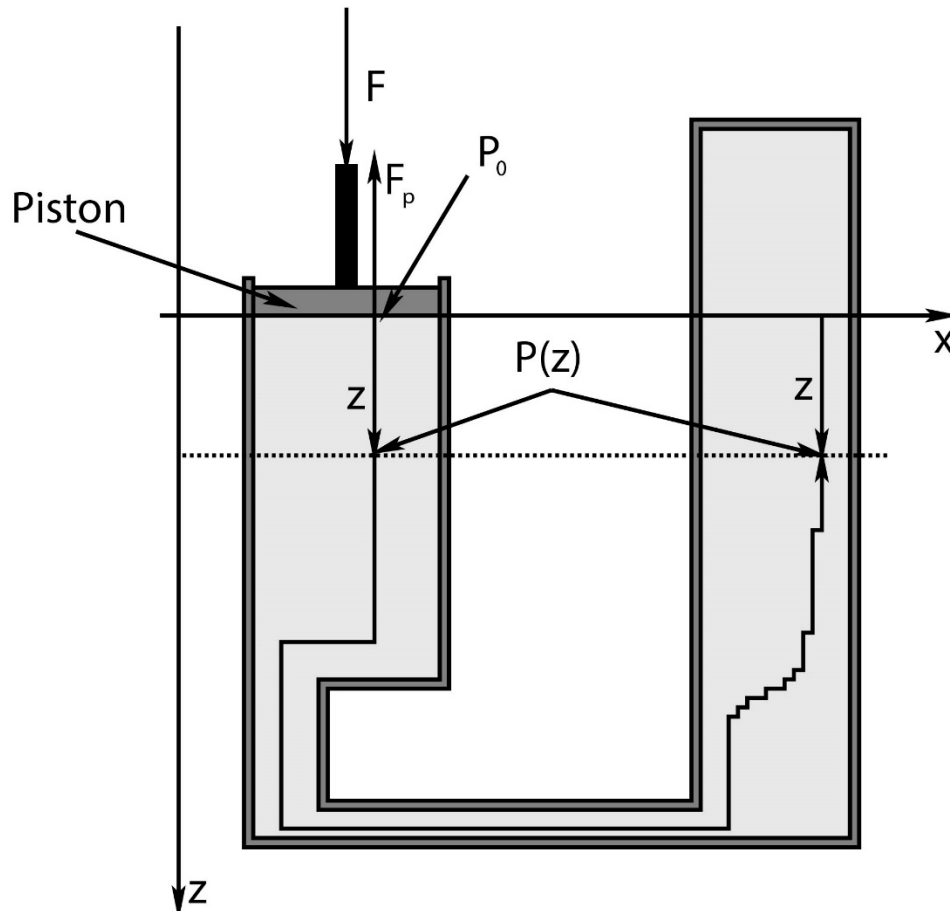
We note that (from the above) the general form of P of a fluid confined to a sealed container has the most general form:

$$P(z) = P_0 + \int_0^z \rho g dz$$

where P_0 is the constant of integration or value of the pressure at the reference **depth** $z = 0$. This has an important consequence that forms the basis of *hydraulics*.

Lecture 5. Fluids

Pascal's Principle and Hydraulics

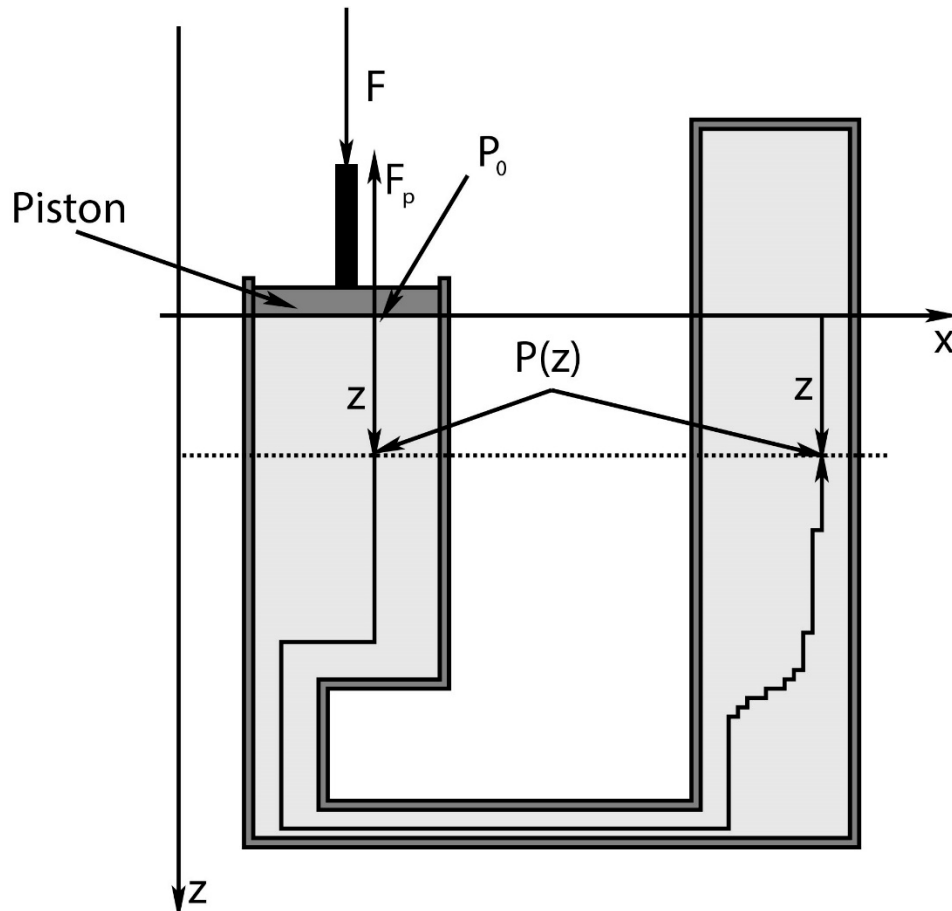


Suppose, that we have an *incompressible fluid* e.g. water confined within a sealed container by e.g. a *piston* that can be pushed or pulled on to *increase or decrease* the *confinement pressure* on the surface of the piston.



Lecture 5. Fluids

Pascal's Principle and Hydraulics



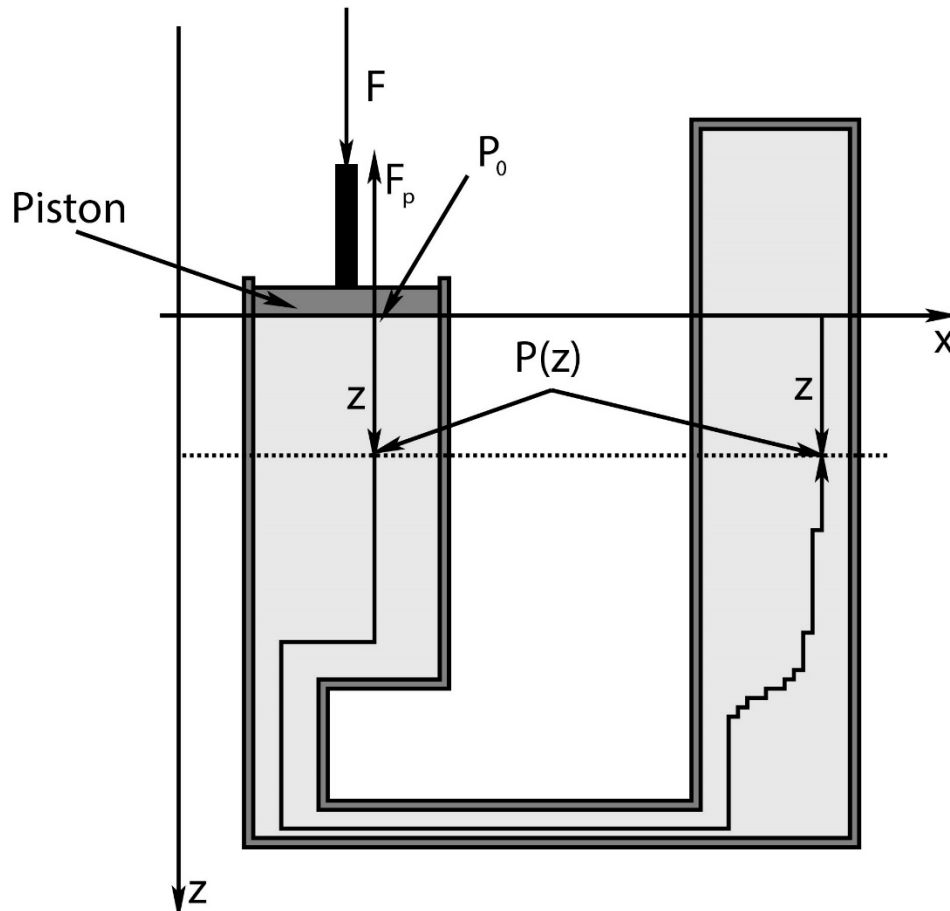
We can push down (or pull back) on the piston with any total downward force F that we like that leaves the system in equilibrium. Since the piston itself is in static equilibrium, the force we push with must be opposed by the pressure in the fluid, which exerts an equal and opposite upwards force:

$$F = F_p = P_0 A$$

where A is the cross sectional area of the piston and where we've put the cylinder face at $z = 0$, which we are obviously free to do.

Lecture 5. Fluids

Pascal's Principle and Hydraulics



The pressure at a depth z in the container is then

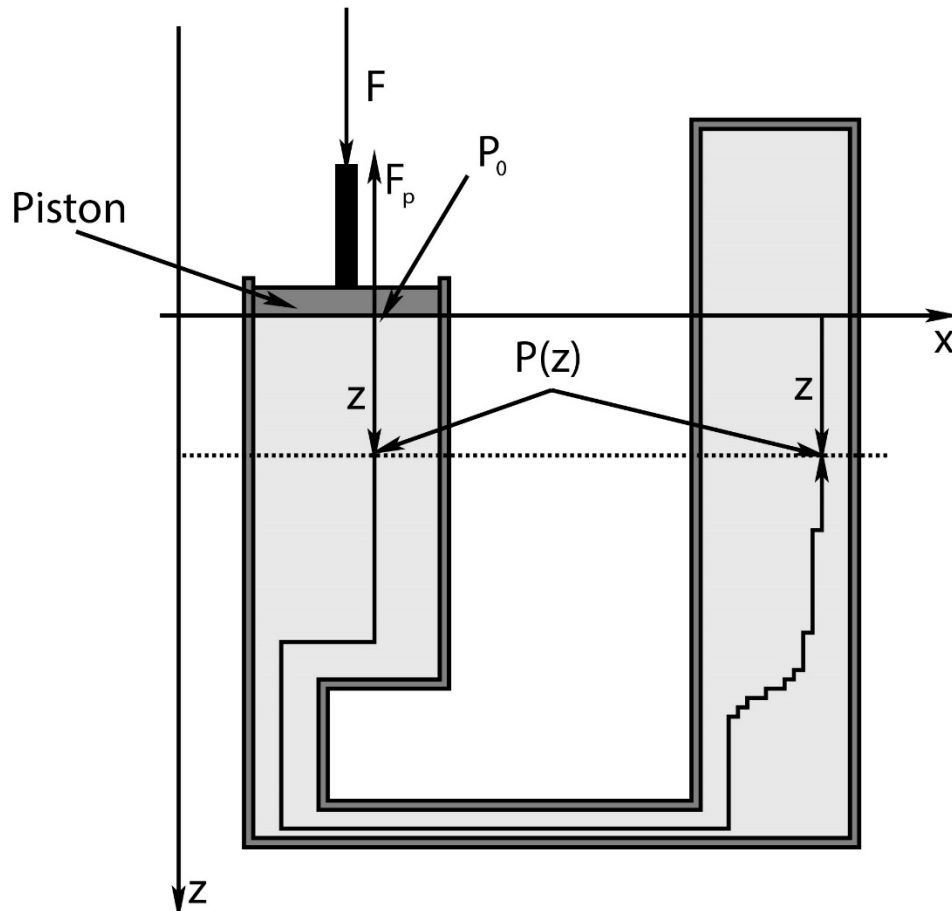
$$P(z) = P_0 + \rho g z$$

where A is the cross sectional area of the piston and where we've put the cylinder face at $z = 0$, which we are obviously free to do.

where $\rho = \rho_w$ if the cylinder is indeed filled with water, but the cylinder could equally well be filled with hydraulic fluid

Lecture 5. Fluids

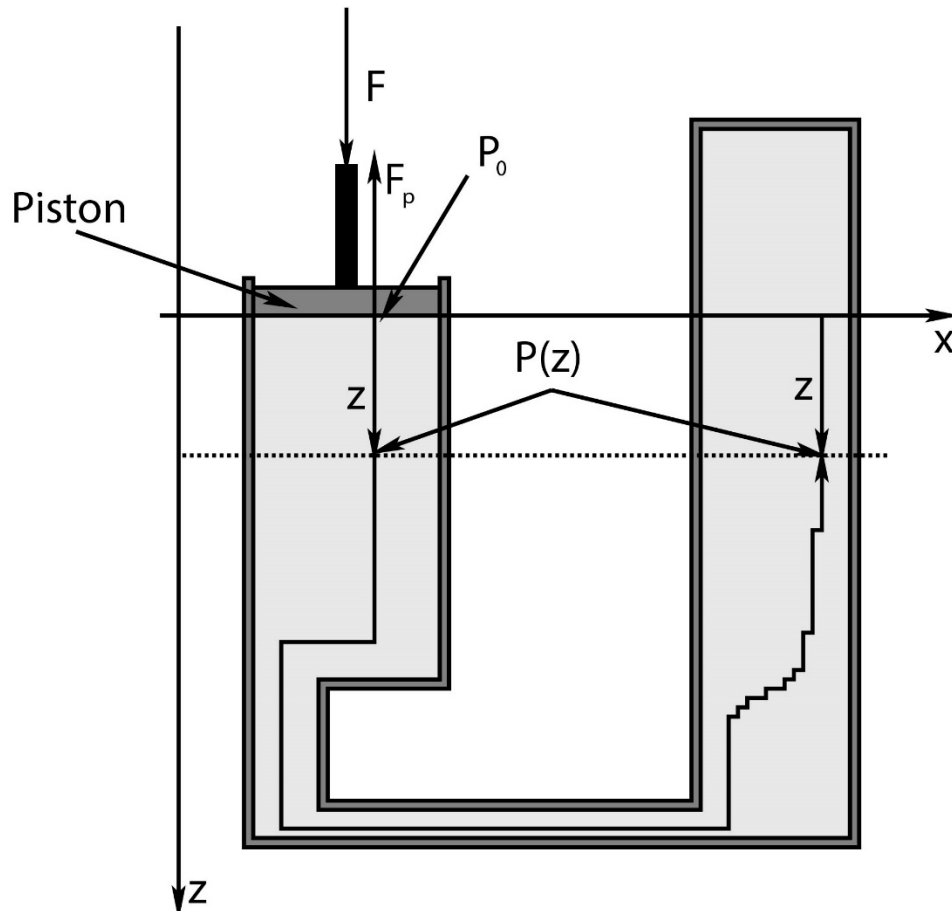
Pascal's Principle and Hydraulics



We recall that the pressure changes only when we change our depth. Moving laterally does not change the pressure, because e.g. $dP/dx = dP/dy = 0$. We can always find a path consisting of vertical and lateral displacements from $z = 0$ to any other point in the container – two such points at the same depth z are shown in figure, along with a vertical/horizontal path connecting them. Clearly these two points *must have the same pressure $P(z)$* !

Lecture 5. Fluids

Pascal's Principle and Hydraulics



Now consider the following. Suppose we start with pressure P_0 (so that the pressure at these two points is $P(z)$), but then change F to make the pressure P'_0 and the pressure at the two points $P'(z)$. Then:

$$P(z) = P_0 + \rho g z$$

$$P'(z) = P'_0 + \rho g z$$

$$\Delta P(z) = P'(z) - P(z) = P'_0 - P_0 = \Delta P_0$$

That is, the pressure change at depth z does not depend on z *at any point in the fluid!* It depends *only on the change in the pressure exerted by the piston!*



Lecture 5. Fluids

Pascal's Principle and Hydraulics

This result is known as *Pascal's Principle* and it holds (more or less) for any compressible fluid, not just incompressible ones, but in the case of compressible fluids the piston will move up or down or in or out and the density of the fluid will change and hence the treatment of the integral will be too complicated to cope with. Pascal's Principle is more commonly given in *English words* as:

Any change in the pressure exerted at a given point on a confined fluid is transmitted, undiminished, throughout the fluid.

Pascal's principle is the basis of *hydraulics*. Hydraulics are a kind of fluid-based simple machine that can be used to greatly amplify an applied force.



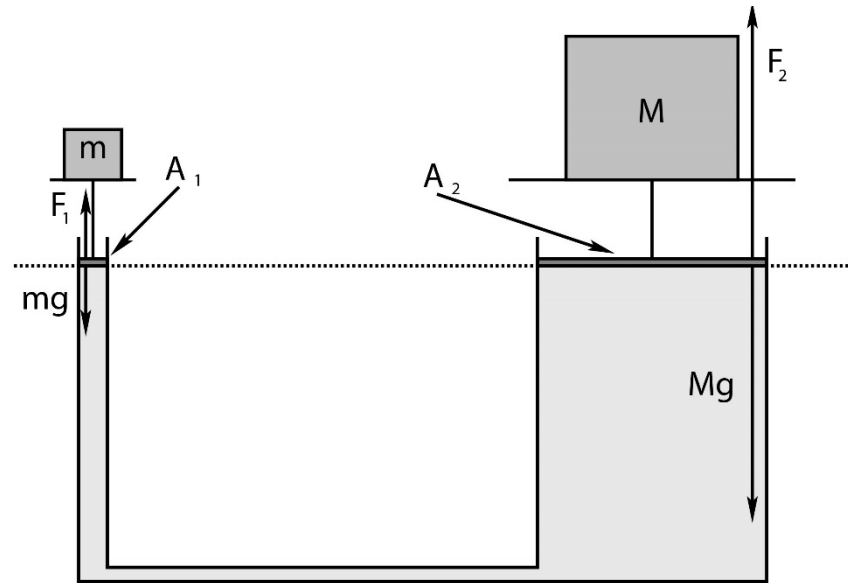
Lecture 5. Fluids

A Hydraulic Lift

Figure illustrates the way we can multiply forces using Pascal's Principle.

Two pistons seal off a pair of cylinders connected by a closed tube that contains an incompressible fluid. The two pistons are deliberately given the same height (which might as well be $z = 0$), then, in the figure, although we could easily deal with the variation of pressure associated with them being at different heights since we know $P(z) = P_0 + \rho gz$.

The two pistons have cross sectional areas A_1 and A_2 respectively, and support a small mass m on the left and large mass M on the right in static equilibrium.





Lecture 5. Fluids

A Hydraulic Lift

For them to be in equilibrium, clearly:

$$F_1 - mg = 0$$

$$F_2 - Mg = 0$$

We also/therefore have:

$$F_1 = P_0 A_1 = mg$$

$$F_2 = P_0 A_2 = Mg$$

Thus

$$\frac{F_1}{A_1} = P_0 = \frac{F_2}{A_2}$$

or (substituting and cancelling g):

$$M = \frac{A_2}{A_1} m$$

A small mass on a small-area piston can easily balance a much larger mass on an equally larger area piston!



Lecture 5. Fluids

A Hydraulic Lift

$$M = \frac{A_2}{A_1} m$$

If we try to lift (say) a car with a hydraulic lift, we have to move the same volume $\Delta V = A\Delta z$ from under the small piston (as it descends) to under the large one (as it ascends). If the small one goes down a distance z_1 and the large one goes up a distance z_2 , then:

$$\frac{z_1}{z_2} = \frac{A_2}{A_1}$$

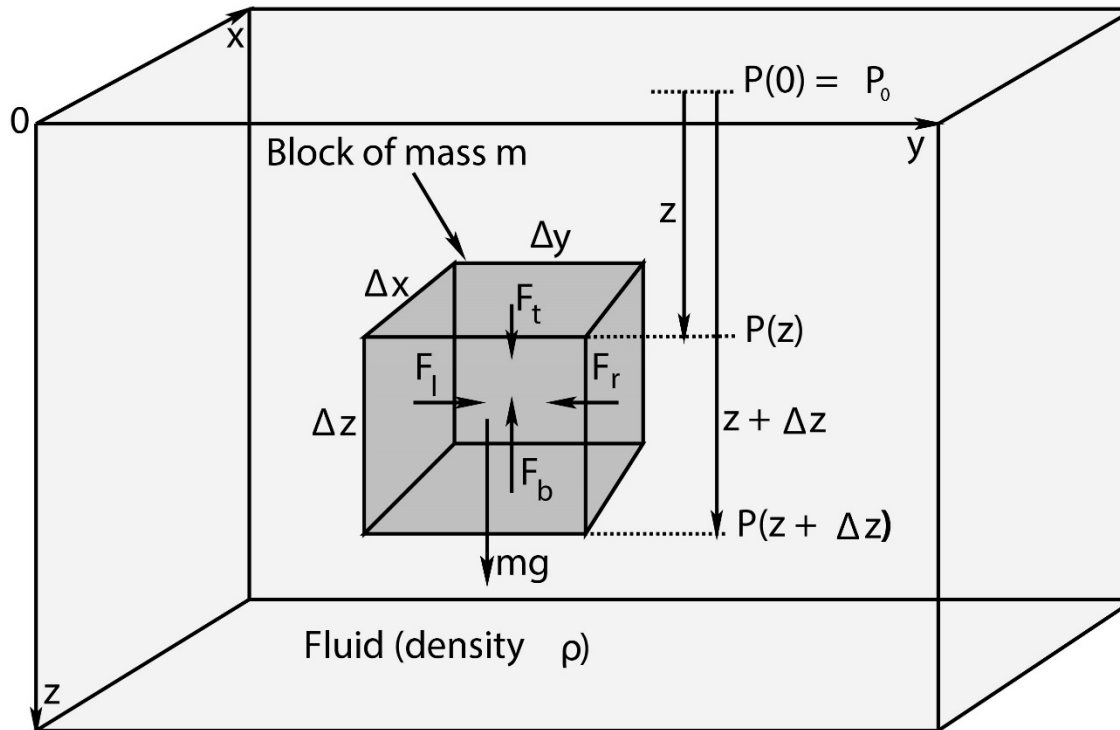
The work done by the two cylinders thus precisely balances:

$$W_2 = F_2 z_2 = F_1 \frac{A_2}{A_1} z_2 = F_1 \frac{A_2}{A_1} z_1 \frac{A_1}{A_2} = F_1 z_1 = W_1$$

The hydraulic arrangement thus transforms pushing a small force through a large distance into a large force moved through a small distance so that the work done *on* piston 1 matches the work done *by* piston 2.

Lecture 5. Fluids

Archimedes' Principle



A solid chunk of “stuff” of mass m and the dimensions shown is immersed in a fluid of density ρ at a depth z . The vertical pressure difference in the fluid (that arises as the fluid itself becomes static static) exerts a vertical force on the cube.



Lecture 5. Fluids

Archimedes' Principle

The net upward force exerted by the fluid is called the **buoyant force** F_b and is equal to:

$$\begin{aligned} F_b &= P(z + \Delta z)\Delta x\Delta y - P(z)\Delta x\Delta y = \\ &= \left((P_0 + \rho g(z + \Delta z)) - (P_0 + \rho g z) \right) \Delta x\Delta y = \\ &= \rho g \Delta z \Delta x \Delta y = \\ &= \rho g \Delta V \end{aligned}$$

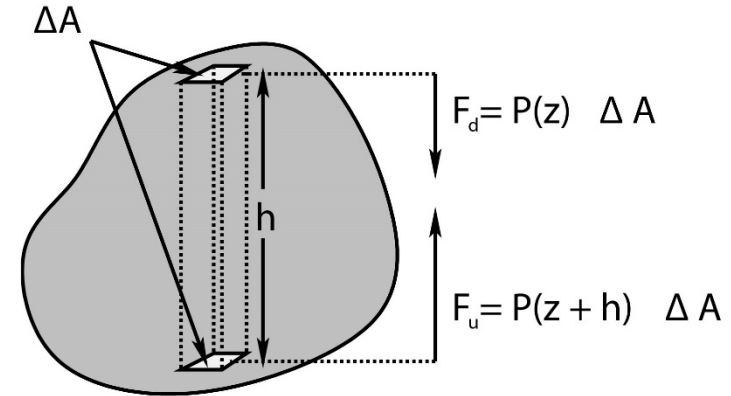
where ΔV is the volume of the small block.

The buoyant force is thus the **weight of the fluid displaced** by this single tiny block. This is all we need to show that the same thing is true for an *arbitrary* immersed shape of object.

Lecture 5. Fluids

Archimedes' Principle

In figure, an arbitrary blob-shape is immersed in a fluid of density ρ . Imagine that we've taken a french-fry cutter and cuts the whole blob into nice rectangular segments, one of which (of length h and cross-sectional area ΔA) is shown. We can trim or average the end caps so that they are all perfectly horizontal by making all of the rectangles arbitrarily small (in fact, differentially small in a moment). In that case the *vertical* force exerted by the fluid on just the two lightly shaded surfaces shown would be:



$$\Delta F_b = \rho g h \Delta A = \rho g \Delta V \text{ (up)}$$

$$F_d = P(z) \Delta A$$

$$F_u = P(z+h) \Delta A$$

where we assume the upper surface is at depth z . Since $P(z+h) = P(z) + \rho g h$, we can find the net upward buoyant force exerted on this little cross-section by subtracting the first from the second:

$$\Delta F_b = F_u - F_d = \rho g h \Delta A = \rho g \Delta V \text{ where the volume of this piece is } \Delta V = h \Delta A.$$



Lecture 5. Fluids

Archimedes' Principle

$$\Delta F_b = F_u - F_d = \rho g h \Delta A = \rho g \Delta V \quad \text{where the volume of this piece is } \Delta V = h \Delta A.$$

We can now let $\Delta A \rightarrow dA$, so that $\Delta V \rightarrow dV$, and write

$$F_b = \int F_b = \int_{V \text{ of blob}} \rho g dV = \rho g V = m_f g$$

where $m_f = \rho V$ is the mass of the fluid displaced, so that $m_f g$ is its weight.

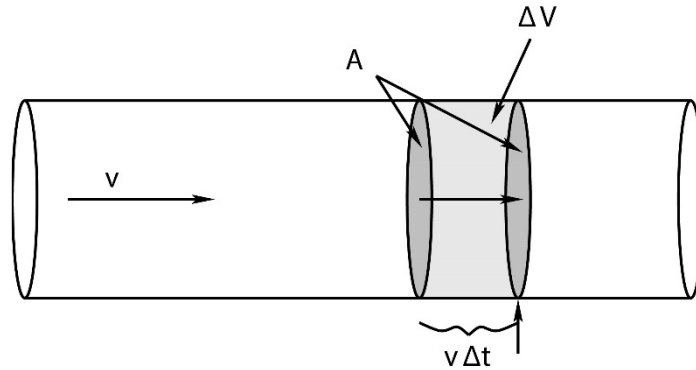
That is:

The total buoyant force on the immersed object is the weight of the fluid displaced by the object.

This statement – in the English or algebraic statement as you prefer – is known as ***Archimedes' Principle***,

Lecture 6. Fluids

Fluid Flow



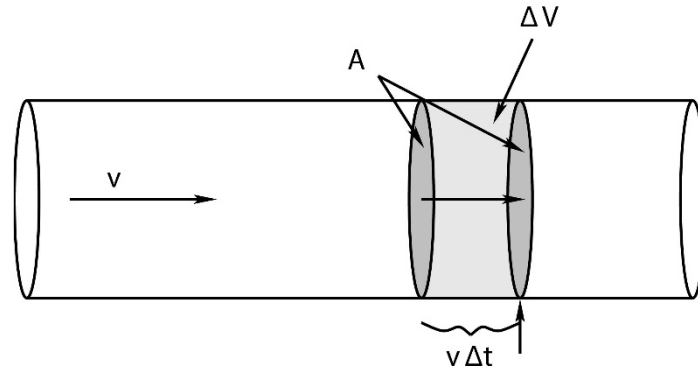
In figure we see fluid flowing from left to right in a circular pipe. The pipe is assumed to be “frictionless” for the time being – to exert no drag force on the fluid flowing within – and hence all of the fluid is moving *uniformly* (at the same speed v with no relative *internal* motion) in a state of *dynamic equilibrium*.

- We are interested in understanding the *flow* or *current* of water carried by the pipe, which we will define to be the *volume per unit time* that passes any given point in the pipe.
- We would like to understand the relationship between area, speed and flow



Lecture 6. Fluids

Fluid Flow



In a time Δt , all of the water within a distance $v\Delta t$ to the left of the second shaded surface will pass *through* this surface and hence past the point indicated by the arrow underneath. The volume of this fluid is just the area of the surface times the height of the cylinder of water:

$$\Delta V = Av\Delta t$$

If we divide out the Δt , we get:

$$I = \frac{\Delta V}{\Delta t} = Av$$

This, then is the *flow*, or *volumetric current* of fluid in the pipe.

Lecture 6. Fluids

Conservation of Flow

Fluid does not, of course, only flow in smooth pipes with a single cross-sectional area. Sometimes it flows from large pipes into smaller ones or vice versa.

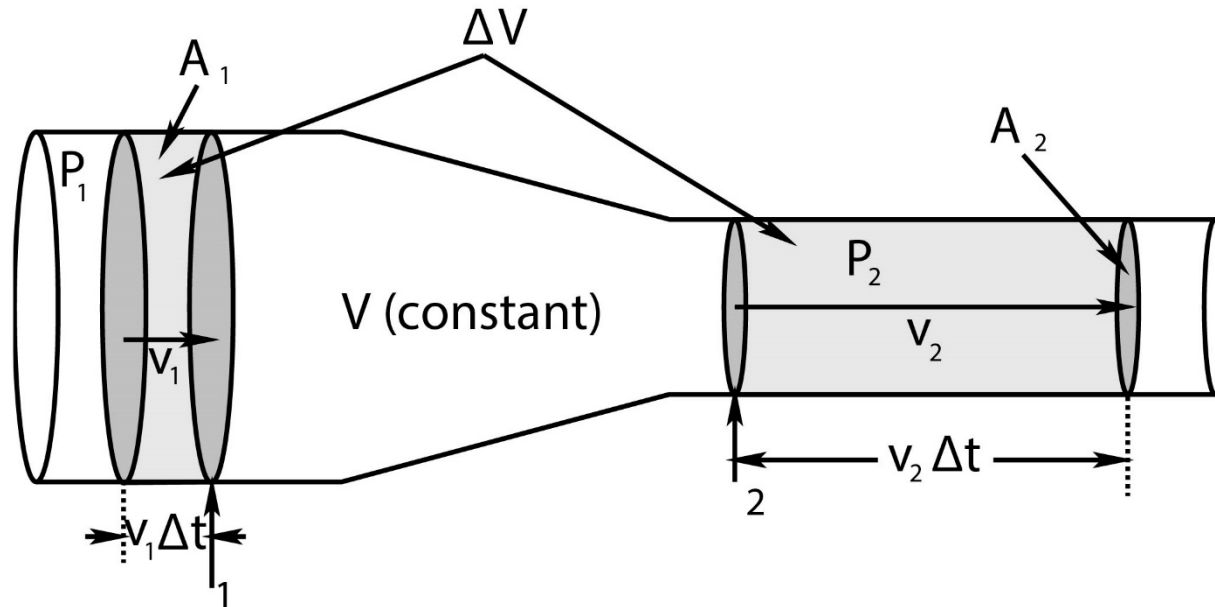
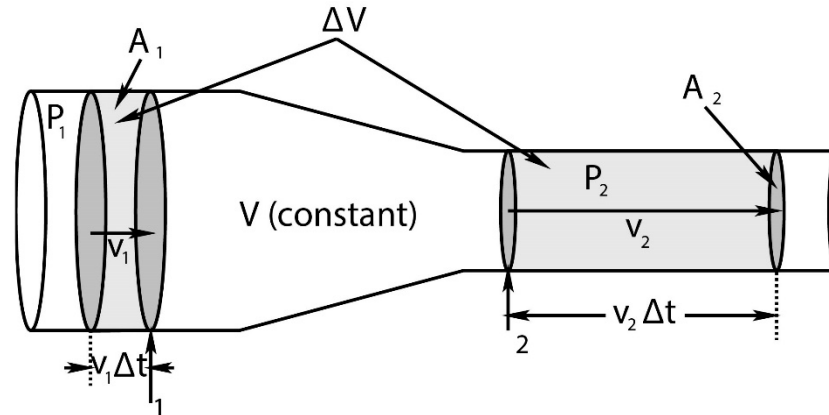


Figure shows a fluid as it flows from just such a wider pipe down a gently sloping neck into a narrower one. As before, we will ignore drag forces and assume that the flow is as uniform as possible. The pressure, speed of the (presumed incompressible) fluid, and cross sectional area for either pipe are P_1 , v_1 , and A_1 in the wider one and P_2 , v_2 , and A_2 in the narrower one.

Lecture 6. Fluids

Conservation of Flow



In a time Δt a volume of fluid $\Delta V = A_1 v_1 \Delta t$ passes through the surface/past the point 1 marked with an arrow in the figure. In the volume between this surface and the next grey surface at the point 2 marked with an arrow ***no fluid can build up*** so actual quantity of mass in this volume must be a *constant*.

This is a kind of *conservation law* which, for a continuous fluid or similar medium, is called a *continuity equation*.

$$\Delta V = A_1 v_1 \Delta t = A_2 v_2 \Delta t$$

$$I = \frac{\Delta V}{\Delta t} = A_1 v_1 = A_2 v_2$$

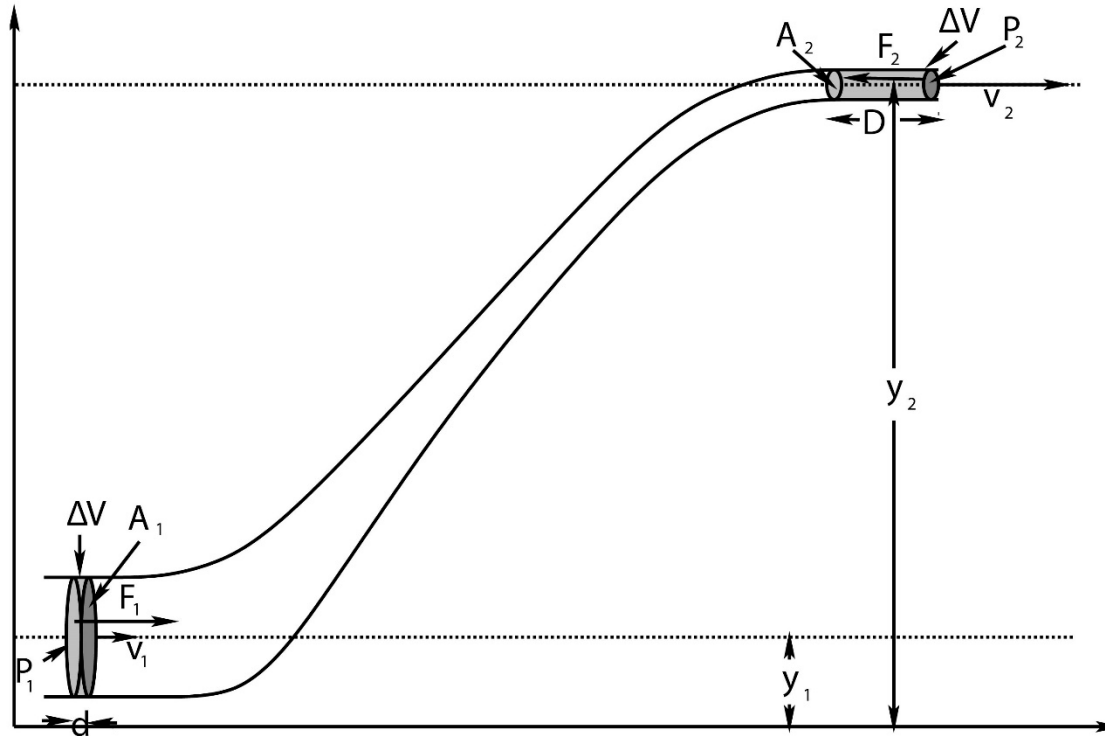
Thus the current or flow through the two surfaces marked 1 and 2 must be the same:

$$A_1 v_1 = A_2 v_2$$



Lecture 6. Fluids

Work-Mechanical Energy in Fluids: Bernoulli's Equation



A circular cross-sectional necked pipe is arranged so that the pipe *changes height* between the larger and smaller sections. We will assume that both pipe segments are narrow compared to the height change, so that we don't have to account for a potential energy difference (per unit volume) between water flowing at the top of a pipe compared to the bottom, but for ease of viewing we do not draw the *picture* that way.



Lecture 6. Fluids

Work-Mechanical Energy in Fluids: Bernoulli's Equation

The fluid is incompressible and the pipe itself does not leak, so fluid cannot build up between the bottom and the top. As the fluid on the bottom moves to the left a distance d (which might be $v_1 \Delta t$ but we don't insist on it as rates will not be important in our result) exactly the same amount fluid must move to the left a distance D up at the top so that fluid is conserved.

The total *mechanical* consequence of this movement is thus the disappearance of a chunk of fluid mass:

$$\Delta m = \rho \Delta V = \rho A_1 d = \rho A_2 D$$

that is moving at speed v_1 and at height y_1 at the bottom and it's appearance moving at speed v_2 and at height y_2 at the top. Clearly **both** the kinetic energy **and** the potential energy of this chunk of mass have changed.



Lecture 6. Fluids

Work-Mechanical Energy in Fluids: Bernoulli's Equation

What caused this change in mechanical energy?

Well, it can only be work.

What does the work?

The walls of the (frictionless, drag free) pipe can do no work as the only force it exerts is perpendicular to the wall and hence to \vec{v} in the fluid.

The only thing left is the ***pressure*** that acts on the entire block of water between the first surface (lightly shaded) drawn at both the top and the bottom as it moves forward to become the second surface (darkly shaded) drawn at the top and the bottom, effecting this net transfer of mass Δm .



Lecture 6. Fluids

Work-Mechanical Energy in Fluids: Bernoulli's Equation

The force F_1 exerted to the right on this block of fluid at the bottom is just $F_1 = P_1 A_1$; the force F_2 exerted to the left on this block of fluid at the top is similarly $F_2 = P_2 A_2$. The work done by the pressure acting over a distance d at the bottom is $W_1 = P_1 A_1 d$, at the top it is $W_2 = -P_2 A_2 D$. The total work is equal to the total change in mechanical energy of the chunk Δm :

$$W_1 + W_2 = E_{mech}(final) - E_{mech}(initial) \quad W_{tot} = \Delta E_{mech}$$

$$P_1 A_1 d - P_2 A_2 D = \left(\frac{1}{2} m v_2^2 + \Delta m g y_2 \right) - \left(\frac{1}{2} m v_1^2 + \Delta m g y_1 \right)$$

$$(P_1 - P_2) \Delta V = \left(\frac{1}{2} \rho \Delta V v_2^2 + \rho \Delta V g y_2 \right) - \left(\frac{1}{2} \rho \Delta V v_1^2 + \rho \Delta V g y_1 \right)$$

$$(P_1 - P_2) = \left(\frac{1}{2} \rho v_2^2 + \rho g y_2 \right) - \left(\frac{1}{2} \rho v_1^2 + \rho g y_1 \right)$$

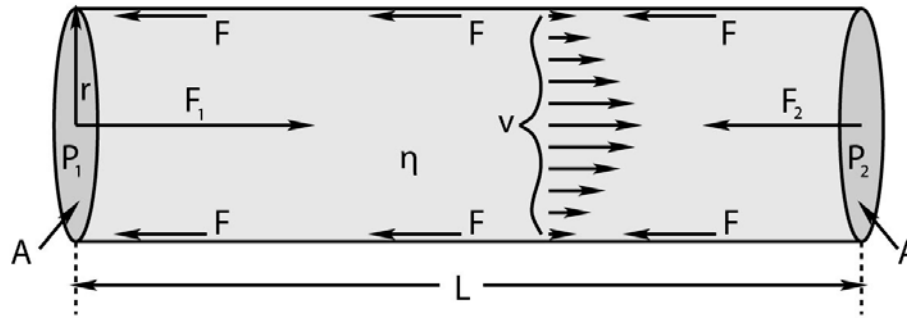
$$P_1 + \frac{1}{2} \rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2} \rho v_2^2 + \rho g y_2 = \text{a constant (units of pressure)}$$

This result is known as ***Bernoulli's Principle***

Lecture 6. Fluids

Fluid Viscosity and Resistance

In the discussion above, we have consistently ignored viscosity and drag, which behave like “friction”, exerting a force parallel to the confining walls of the pipe in the opposite direction to the relative motion of fluid and pipe.

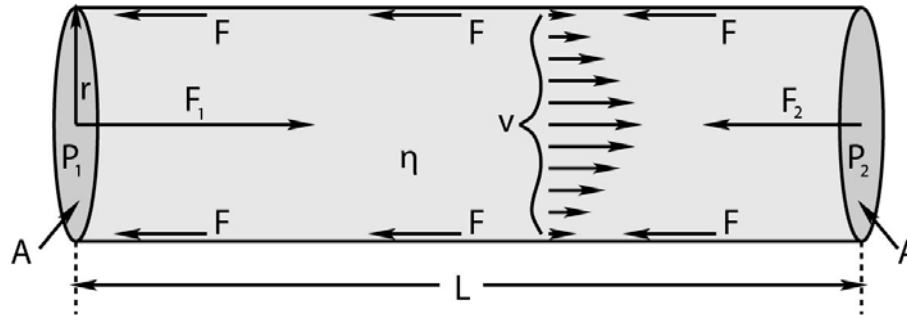


In figure a circular pipe is carrying a fluid with viscosity μ from left to right *at a constant speed*. Once again, this is a sort of dynamic equilibrium; the net force on the fluid in the pipe segment shown must be zero for the speed of the fluid through it to be maintained unabated during the flow.

The fluid is in contact with and interacts with the walls of the pipe, creating a thin layer of fluid at least a few atoms thick that are “at rest”, stuck to the pipe. As fluid is pushed through the pipe, this layer at rest interacts with and exerts an *opposing force* on the layer moving just above it via the viscosity of the fluid. This layer in turn interacts with and slows the layer above it and so on right up to the center of the pipe, where the fluid flows most rapidly.

Lecture 6. Fluids

Fluid Viscosity and Resistance



The interaction of the surface layer with the fluid, redistributed to the whole fluid via the viscosity, exerts a net **opposing force** on the fluid as it moves through the pipe. In order for the average speed of the fluid to continue, an outside force must act on it with an equal and opposite force. The only available source of this force in the figure is obviously the **fluid pressure**; if it is larger on the left than on the right (as shown) it will exert a net force on the fluid in between that can balance the drag force exerted by the walls.

The forces at the ends are $F_1 = P_1 A$, $F_2 = P_2 A$. The net force acting on the fluid mass is thus:

$$\Delta F = F_1 - F_2 = (P_1 - P_2)A$$

All things being equal, we expect the flow rate to increase linearly with v , and for *laminar* flow, the drag force is proportional to v . Therefore we expect that:

$$\Delta F = F_d \propto v \propto I \text{ (the flow)}$$



Lecture 6. Fluids

Fluid Viscosity and Resistance

We can then divide out the area and write:

$$\Delta P \propto \frac{I}{A}$$

We cannot derive the constant of proportionality in this expression, and we will omit some math and just write following result:

$$\Delta P = I \left(\frac{8L\mu}{\pi r^4} \right) = IR$$

where I have introduced the resistance of the pipe to flow:

$$R = \frac{8L\mu}{\pi r^4}$$

This equation is known as **Poiseuille's Law** and is a key relation for physicians and plumbers to know because it describes both flow of water in pipes and the flow of blood in blood vessels wherever the flow is slow enough that it is laminar and not turbulent



Lecture 6. Fluids

A Brief Note on Turbulence

The velocity of the flow in a circular pipe (and other parameters such as μ and r) can be transformed into a general dimensionless parameter called the **Reynolds Number (Re)**.

The Reynolds number for a circular pipe is:

$$Re = \frac{\rho v D}{\mu} = \frac{\rho v 2r}{\mu}$$

where $D = 2r$ is the **hydraulic diameter**, which in the case of a circular pipe is the actual diameter.

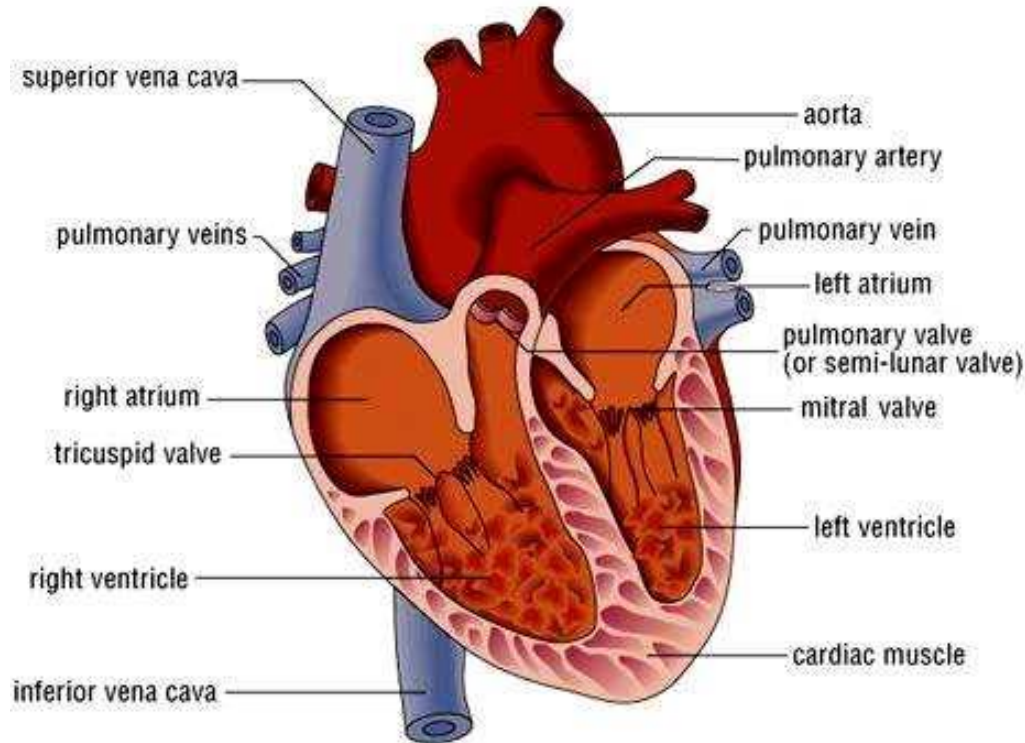
The one thing the Reynolds number does for us is that it serves as a **marker for the transition to turbulent flow**.

For $Re < 2300$ flow in a circular pipe is laminar and all of the relations above hold.

Turbulent flow occurs for $Re > 4000$. In between is the region known as the **onset of turbulence**, where the **resistance of the pipe depends on flow in a very nonlinear fashion**, and among other things **dramatically increases** with the Reynolds number.

Lecture 6. Fluids

The Human Circulatory System



Here is a list of True Facts about the human cardiovascular system:

- The heart, illustrated in the schematic in figure is the “*pump*” that drives blood through your blood vessels.



Lecture 6. Fluids

The Human Circulatory System

- **The blood vessels are differentiated into three distinct types:**

- **Arteries**, which lead strictly *away from the heart* and which contain a muscular layer that elastically dilates and contracts the arteries in a synchronous way to help carry the surging waves of blood. This acts as a “shock absorber” and hence reduces the peak systolic blood pressure. Arteries split up the farther one is from the heart, eventually becoming **arterioles**, the very small arteries that actually split off into capillaries.
- **Capillaries**, which are a dense network of very fine vessels (often only a single cell thick) that *deliver oxygenated blood throughout all living tissue* so that the oxygen can disassociate from the carrying hemoglobin molecules and diffuse into the surrounding cells in systemic circulation, or *permit the oxygenation of blood* in pulmonary circulation.
- **Veins**, which lead strictly *back to the heart* from the capillaries. Veins also have a muscle layer that expand or contract to aid in thermoregulation and regulation of blood pressure as one lies down or stands up. Veins also provide “capacitance” to the circulatory system and store the body’s “spare” blood; 60% of the body’s total blood supply is usually in the veins at any one time. Most of the veins, especially long vertical veins, are equipped with one-way **venous valves** every 4-9 cm that prevent backflow and pooling in the lower body during e.g. diastoli.

Blood from the capillaries is collected first in **venules** (the return-side equivalent of arterioles) and then into veins proper.



Lecture 6. Fluids

The Human Circulatory System

- There are two distinct circulatory systems in humans (and in the rest of the mammals and birds):
 - ***Systemic circulation***, where oxygenated blood enters the heart via pulmonary veins *from* the lungs and is pumped at high pressure *into* systemic arteries that deliver it through the capillaries and (deoxygenated) back via systemic veins to the heart.
 - ***Pulmonary circulation***, where deoxygenated blood that has returned *from* the system circulation is pumped *into* pulmonary arteries that deliver it to the lungs, where it is oxygenated and returned to the heart by means of pulmonary veins. These two distinct circulations ***do not mix*** and together, ***form a closed double circulation loop***.



Lecture 6. Fluids

The Human Circulatory System

- **Blood pressure** is generally measured and reported in terms of two numbers:
 - **Systolic** blood pressure. This is the *peak/maximum arterial pressure* in the wave pulse generated that drives *systemic circulation*. It is measured in the (brachial artery of the) arm, where it is supposed to be a reasonably accurate reflection of peak aortic pressure just outside of the heart, where, sadly, it cannot easily be directly measured without resorting to invasive methods that are, in fact, used e.g. during surgery.
 - **Diastolic** blood pressure. This is the *trough/minimum arterial pressure* in the wave pulse of systemic circulation.

“Normal” Systolic systemic blood pressure can fairly accurately be estimated on the basis of the distance between the heart and the feet; a distance on the order of 1.5 meters leads to a pressure difference of around 0.15 atm or 120 mmHg.

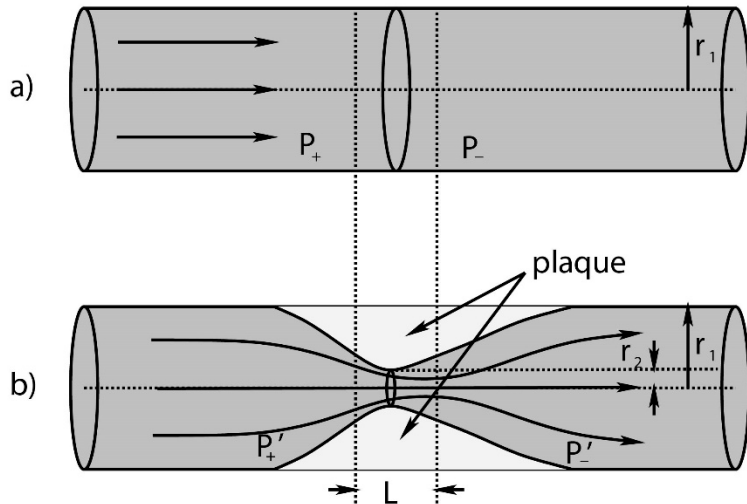
Blood is driven through the relatively high resistance of the capillaries by the *difference* in arterial pressure and venous pressure. The venous system is entirely a *low pressure return*; its peak pressure is typically order of 0.008 bar (6 mmHg). This can be understood and predicted by the mean distance between valves in the venous system – the pressure difference between one valve and another (say) 8 cm higher is approximately $\rho_b g \times 0.08 \approx 0.008$ bar. However, this pressure is not really static – it varies with the delayed pressure wave that causes blood to surge its way up, down, or sideways through the veins on its way back to the atria of the heart.

Lecture 6. Fluids

Atherosclerotic Plaque Partially Occludes a Blood Vessel

Atherosclerosis – granular deposits of fatty material called *plaques* that attach to the walls of e.g. arteries and gradually thicken over time, generally associated with high blood cholesterol and lipidemia. The risk factors for atherosclerosis form a list as long as your arm and its fundamental causes are not well understood, although they are currently believed to form as an inflammatory response to surplus low density lipoproteins (one kind of cholesterol) in the blood.

In figure two arteries are illustrated.

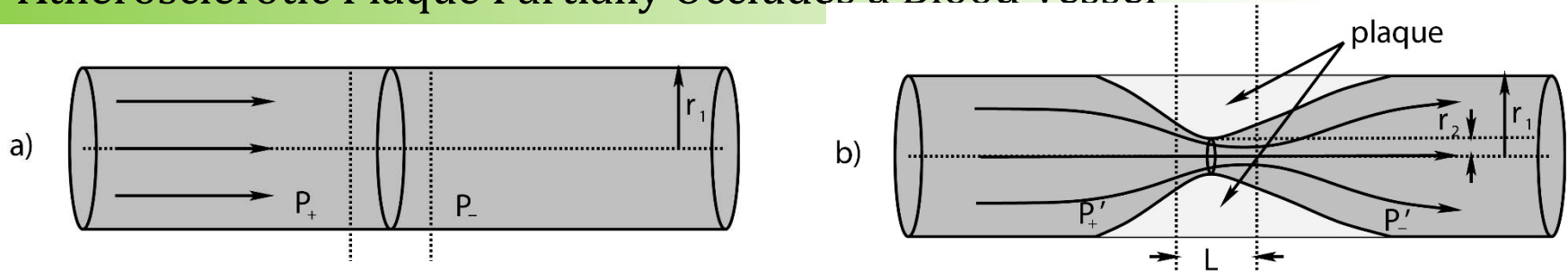


Artery a) is “clean”, has a radius of r_1 , and (from the Poiseuille Equation above) has a very low resistance to any given flow of blood. Because R_a over the length L is low, there is very little pressure drop between P_+ and P_- on the two sides of any given stretch of length L . The velocity profile of the fluid is also more or less uniform in the artery, slowing a bit near the walls but generally moving smoothly throughout the entire cross-section.

Artery b) has a significant deposit of atherosclerotic plaques that have coated the walls and reduced the effective radius of the vessel to $\sim r_2$ over an extended length L . The vessel is perhaps 90% occluded – only 10% of its normal cross-sectional area is available to carry fluid.

Lecture 6. Fluids

Atherosclerotic Plaque Partially Occludes a Blood Vessel



We can now easily understand several things about this situation. First, if the total *flow* in artery b) is still being maintained at close to the levels of the flow in artery a) (so that tissue being oxygenated by blood delivered by this artery is not being critically starved for oxygen yet) the ***fluid velocity in the narrowed region is ten times higher than normal!*** Since the Reynolds number for blood flowing in primary arteries is normally around 1000 to 2000, increasing v by a factor of 10 increases the Reynolds number by a factor of 10, causing the flow to become *turbulent* in the obstruction. This tendency is even more pronounced than this figure suggests – I've drawn a nice symmetric occlusion, but the atheroma (lesion) is more likely to grow predominantly on one side and irregular lesions are more likely to disturb laminar flow even for smaller Reynolds numbers.

This turbulence provides the basis for one method of possible detection and diagnosis – you can *hear* the turbulence (with luck) through the stethoscope during a physical exam. Physicians get a lot of practice listening for turbulence since turbulence produced by *artificially* restricting blood flow in the brachial artery by means of a constricting cuff is basically what one listens for when taking a patient's blood pressure. It really shouldn't be there, especially during diastole, the rest of the time.



Lecture 7. Thermodynamics

Thermodynamics

Lecture 7. Thermodynamics

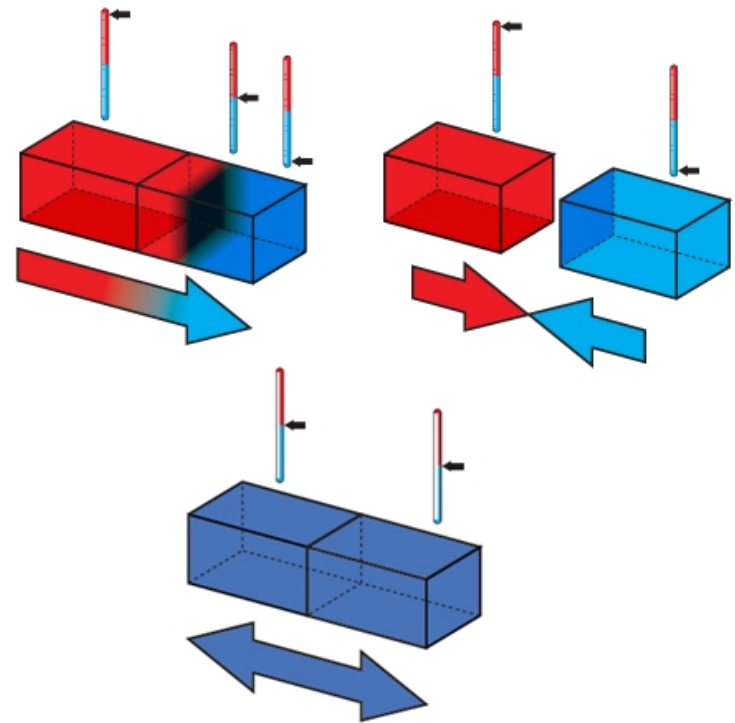
0th Law of Thermodynamics Summary

Thermal Equilibrium

A system with many microscopic components (for example, a gas, a liquid, a solid with many molecules) that is isolated from all forms of energy exchange and left alone for a “long time” moves toward a state of *thermal equilibrium*.

A system in thermal equilibrium is characterized by a set of macroscopic quantities that depend on the system in question and characterize its “state” (such as pressure, volume, density) that do not change in time.

Two systems are said to be in (mutual) thermal equilibrium if, when they are placed in “thermal contact” (basically, contact that permits the exchange of energy between them), their state variables do not change.



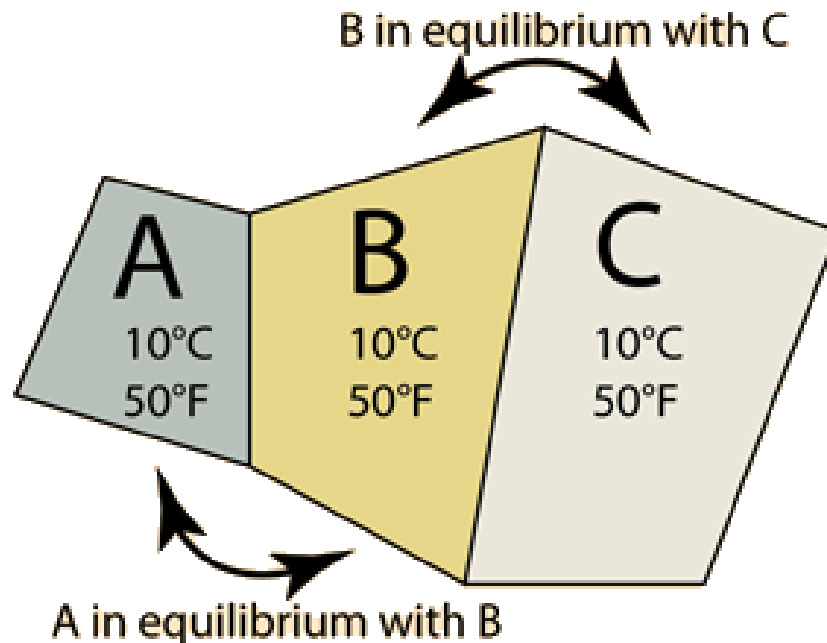


Lecture 7. Thermodynamics

0th Law of Thermodynamics Summary

Zeroth Law of Thermodynamics

If system A is in thermal equilibrium with system C, and system B is in thermal equilibrium with system C, then system A is in thermal equilibrium with system B.

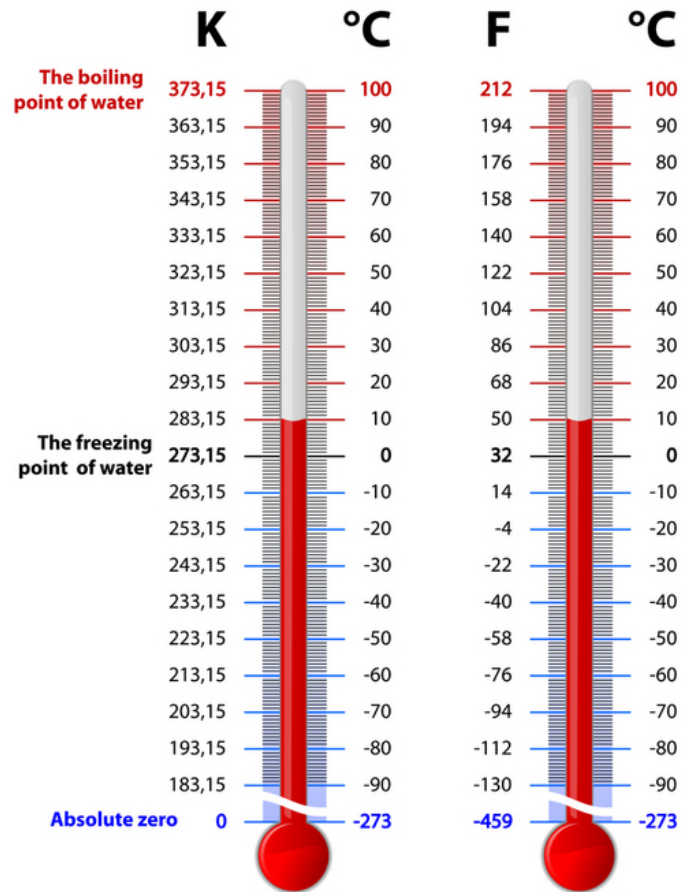


Lecture 7. Thermodynamics

0th Law of Thermodynamics Summary

Temperature Scales

- **Fahrenheit:** This is one of the oldest scales, and is based on the coldest temperature that could be achieved with a mix of ice and alcohol. In it the freezing point of water is at 32°F, the boiling point of water is at 212°F.
- **Celsius or Centigrade:** This is a very sane system, where the freezing point of water is at 0°C and the boiling point is at 100°C. The degree size is thus 9/5 as big as the Fahrenheit degree.
- **Kelvin or Absolute:** 0°K is the lowest possible temperature, where the internal energy of a system is at its absolute minimum. The degree size is the same as that of the Centigrade or Celsius scale. This makes the freezing point of water at atmospheric pressure 273.16°K, the boiling point at 373.16°K.





Lecture 7. Thermodynamics

The First Law of Thermodynamics

Internal Energy

Internal energy is all the mechanical energy in all the components of a system. For example, in a monoatomic gas it might be the sum of the kinetic energies of all the gas atoms. In a solid it might be the sum of the kinetic and potential energies of all the particles that make up the solid.

Heat

Heat is a bit more complicated. It is internal energy as well, but it is internal energy that is *transferred* into or out of a given system. Furthermore, it is in some fundamental sense “disorganized” internal energy – energy with no particular organization, random energy. Heat flows into or out of a system in response to a temperature difference, always flowing from hotter temperature regions (cooling them) to cooler ones (warming them).

Common units of heat include the ever-popular Joule and the *calorie* (the heat required to raise the temperature of 1 gram of water at 14.5°C to 15.5°C. Note that $1 \text{ cal} = 4.186 \text{ J}$).



Lecture 7. Thermodynamics

The First Law of Thermodynamics

Heat Capacity

If one adds heat to an object, its temperature usually increases (exceptions include at a state boundary, for example when a liquid boils). In many cases the temperature change is linear in the amount of heat added. We define the heat capacity C of an object from the relation:.

$$\Delta Q = C\Delta T$$

where Q is the heat that flows into a system to increase its temperature by T .

Many substances have a known heat capacity per unit mass. This permits us to also write:

$$\Delta Q = mC\Delta T$$

where C is the specific heat of a substance. The specific heat of liquid water is approximately:

$$C_{\text{water}} = 1 \frac{\text{calorie}}{\text{kg} \cdot ^\circ\text{C}}$$



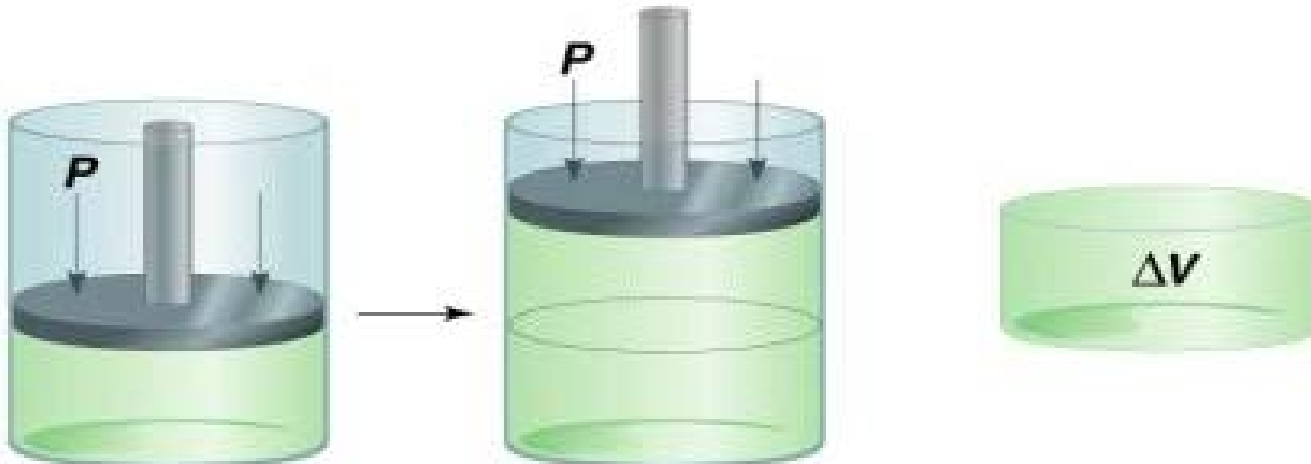
Lecture 7. Thermodynamics

The First Law of Thermodynamics

Work Done by a Gas

$$W = \int_{V_i}^{V_f} P dV$$

This is the area under the $P(V)$ curve, suggesting that we draw lots of state diagrams on a P and V coordinate system. Both heat transfer and work depend on the path a gas takes $P(V)$ moving from one pressure and volume to another.





Lecture 7. Thermodynamics

The First Law of Thermodynamics

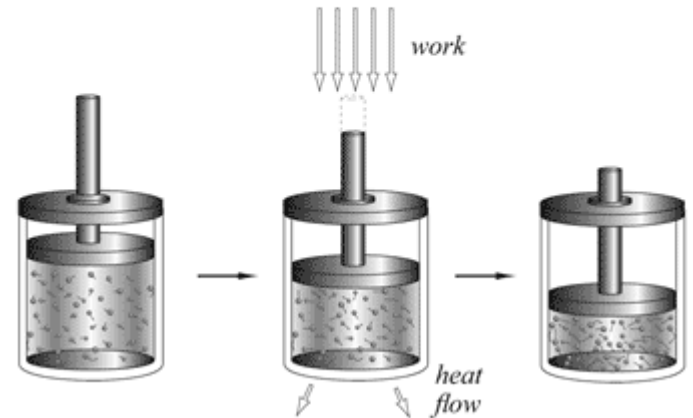
The First Law of Thermodynamics

$$\Delta U = \Delta Q - W$$

In words, this is that the change in total mechanical energy of a system is equal to heat put into the system plus the work done *on* the system (which is minus the work done *by* the system, hence the minus above).

This is just, at long last, the fully *generalized* law of conservation of energy. All the cases where mechanical energy was not conserved in previous chapters because of nonconservative forces, the missing energy appeared as *heat*, energy that naturally flows from hotter systems to cooler ones.

$$\begin{aligned}\Delta U &= Q - W \\ &= Q - P^* \Delta V\end{aligned}$$





Lecture 7. Thermodynamics

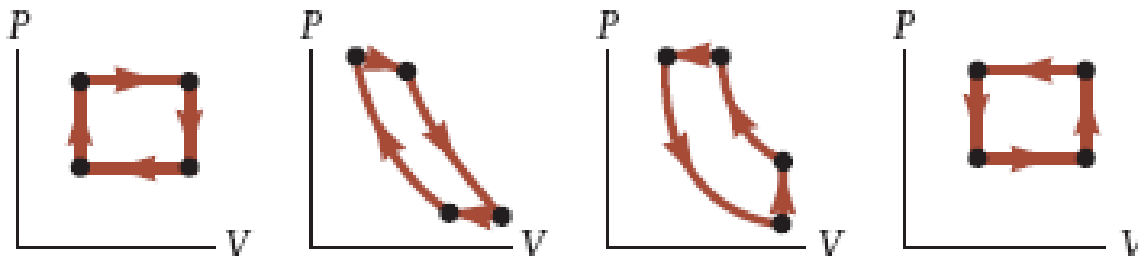
The First Law of Thermodynamics

Cyclic Processes

Most of what we study in these final sections will lead us to an understanding of simple heat engines based on gas expanding in a cylinder and doing work against a piston. In order to build a true engine, the engine has to go around in a repetitive *cycle*. This cycle typically is represented by a closed loop on a state e.g. $P(V)$ curve. A direct consequence of the 1st law is that the ***net work done by the system per cycle is the area inside the loop of the $P(V)$ diagram.*** Since the internal energy is the same at the beginning and the end of the cycle, it also tells us that:

$$\Delta Q_{\text{cycle}} = W_{\text{cycle}}$$

the heat that flows into the system per cycle must exactly equal the work done by the system per cycle.





Lecture 7. Thermodynamics

The First Law of Thermodynamics

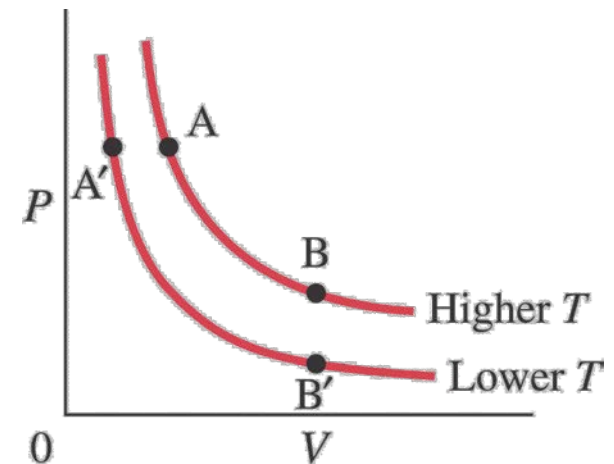
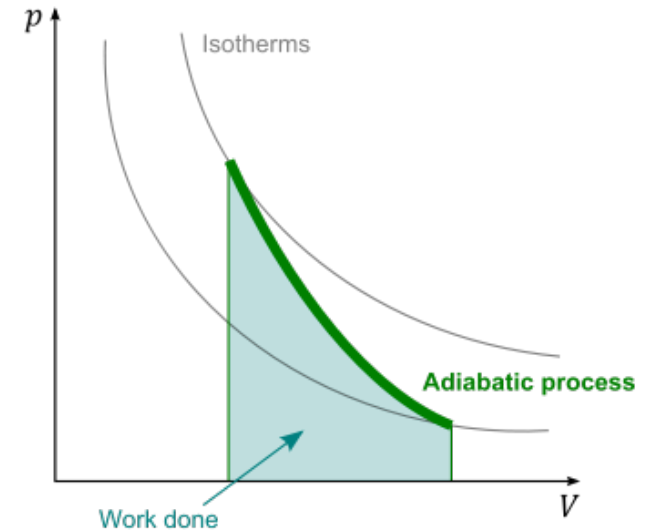
Adiabatic Processes are processes (PV curves) such that no heat enters or leaves an (insulated) system.

The adiabatic condition:

$$PV^\gamma = \text{const}$$

Isothermal Processes are processes where the temperature T of the system remains constant.

$$PV = \text{const}$$



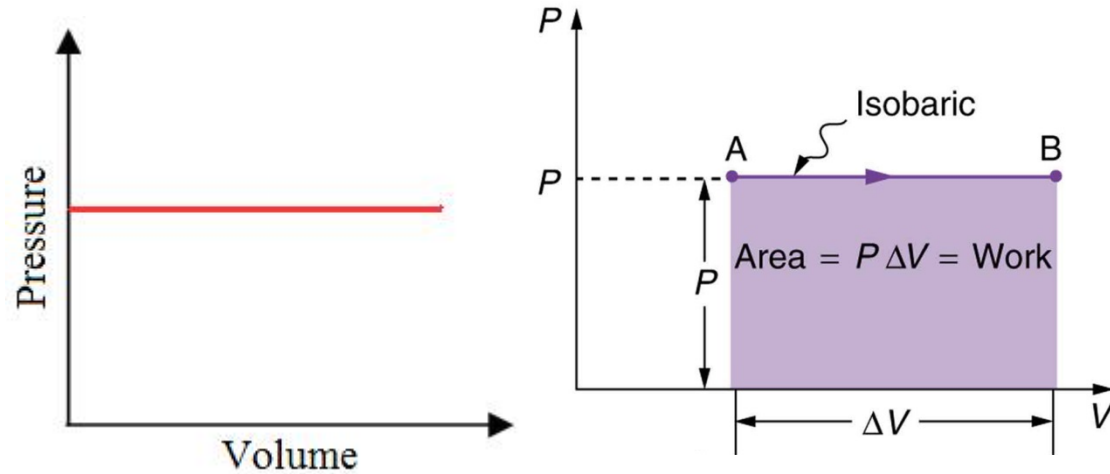


Lecture 7. Thermodynamics

The First Law of Thermodynamics

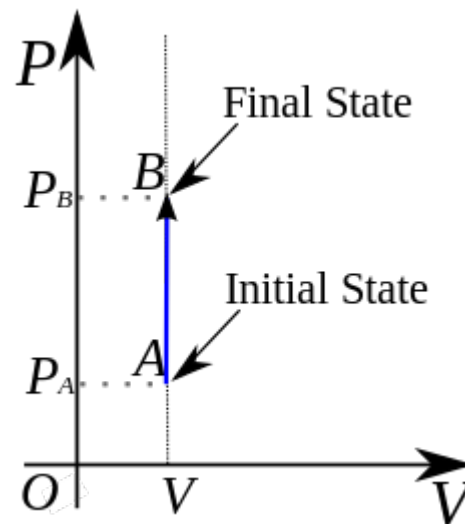
Isobaric Processes are processes that occur at constant pressure.

$$V \sim T$$



Iso volumetric Processes are processes that occur at constant volume

$$P \sim T$$





Lecture 7. Thermodynamics

The First Law of Thermodynamics

Work done by an Ideal Gas:

$$PV = NkT$$

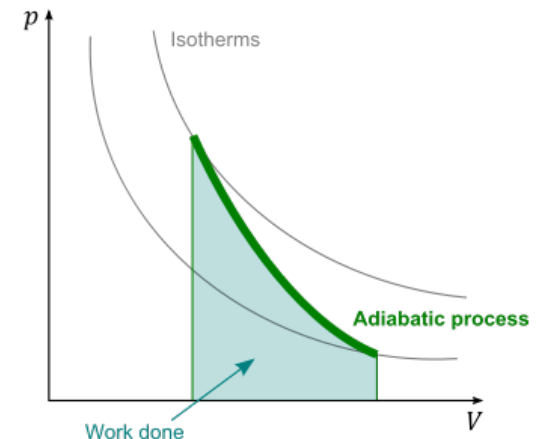
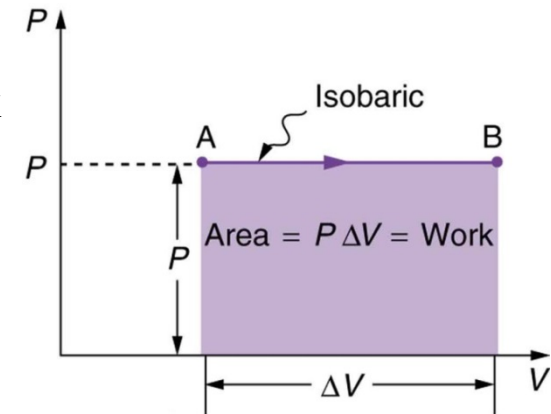
where N is the number of gas atoms or molecules. Isothermal work at (fixed) temperature T_0 is thus:

$$W = \int_{V_1}^{V_2} \frac{NkT_0}{V} dV = NkT \ln \left(\frac{V_2}{V_1} \right)$$

Isobaric work is trivial. $P = P_0$ is a constant, so

$$W = \int_{V_1}^{V_2} P_0 dV = P_0(V_2 - V_1)$$

Adiabatic work is a bit tricky and depends on some of the internal properties of the gas (for example, whether it is mono- or diatomic).



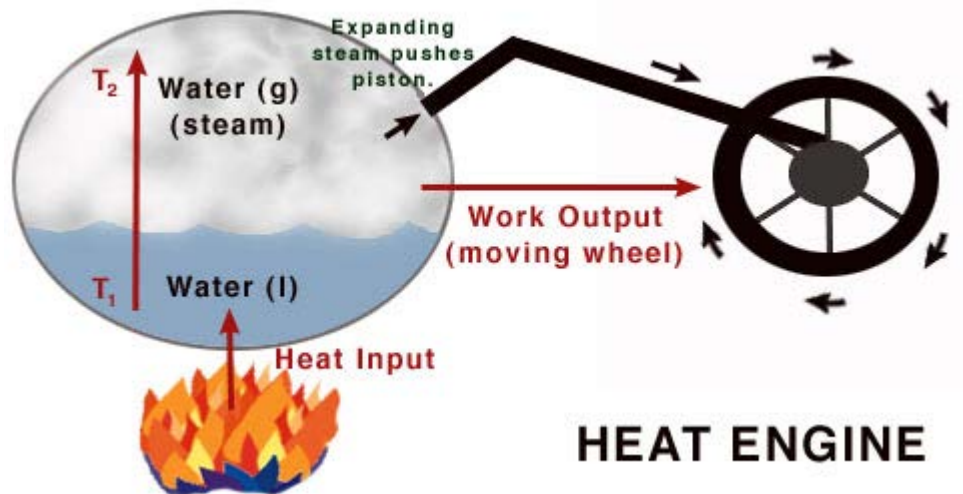
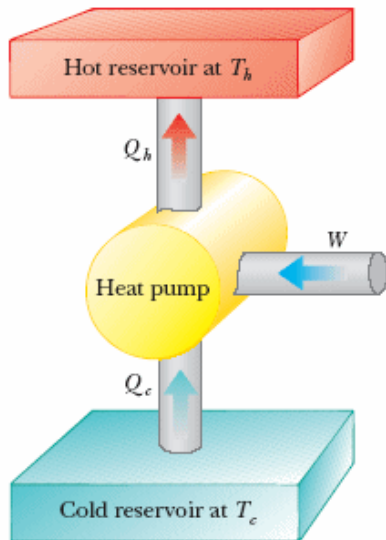
Lecture 7. Thermodynamics

Second Law of Thermodynamics

A **heat engine** is a cyclic device that takes heat Q_H in from a *hot reservoir*, converts *some* of it to work W , and rejects the rest of it Q_C to a *cold reservoir* so that at the end of a cycle it is in the same state (and has the same internal energy) with which it began. The net work done per cycle is the area inside the PV curve.

The *efficiency* of a heat engine is defined to be:

$$\epsilon = \frac{W}{Q_H} = \frac{Q_H - Q_C}{Q_H} = 1 - \frac{Q_C}{Q_H}$$





Lecture 7. Thermodynamics

Kelvin-Planck statement of the Second Law of Thermodynamics

It is impossible to construct a cyclic heat engine that produces no other effect but the absorption of energy from a hot reservoir and the production of an equal amount of work.

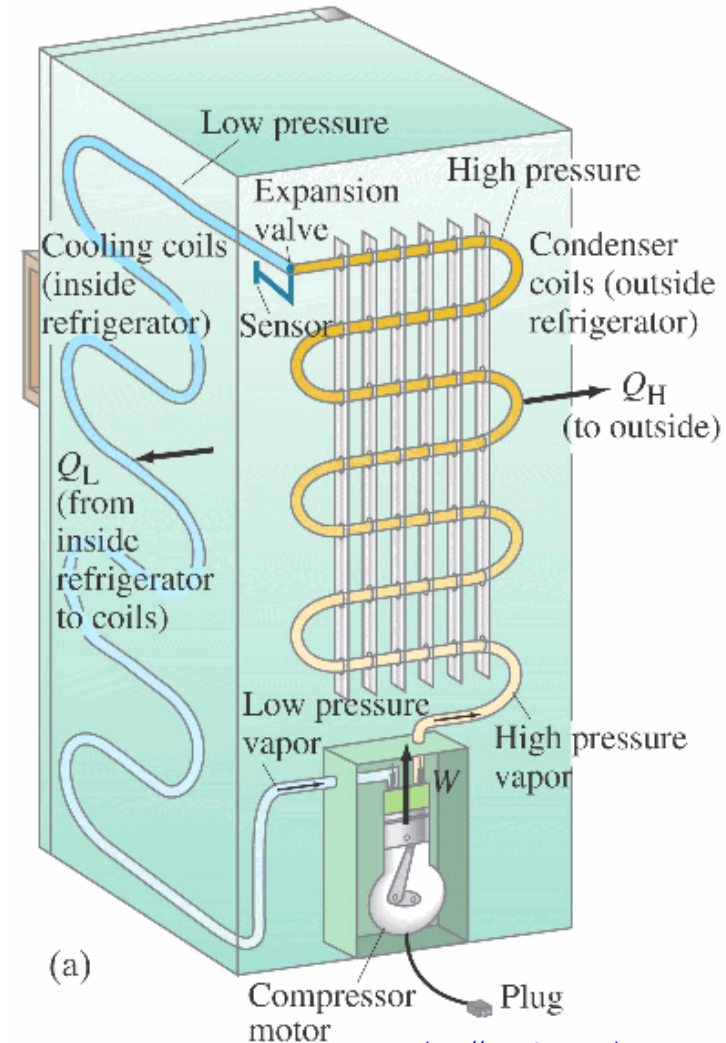
Lecture 7. Thermodynamics

Refrigerators (and Heat Pumps)

A **refrigerator** is basically a cyclic heat engine run backwards. In a cycle it takes heat Q_C in from a cold reservoir, does work W on it, and rejects a heat Q_H to a hot reservoir. Its net effect is thus to make the cold reservoir colder (refrigeration) by removing heat from inside it to the warmer warm reservoir (warming it still further, e.g. as a heat pump).

The **coefficient of performance** of a refrigerator is defined to be

$$\text{COP} = \frac{Q_C}{W}$$





Lecture 7. Thermodynamics

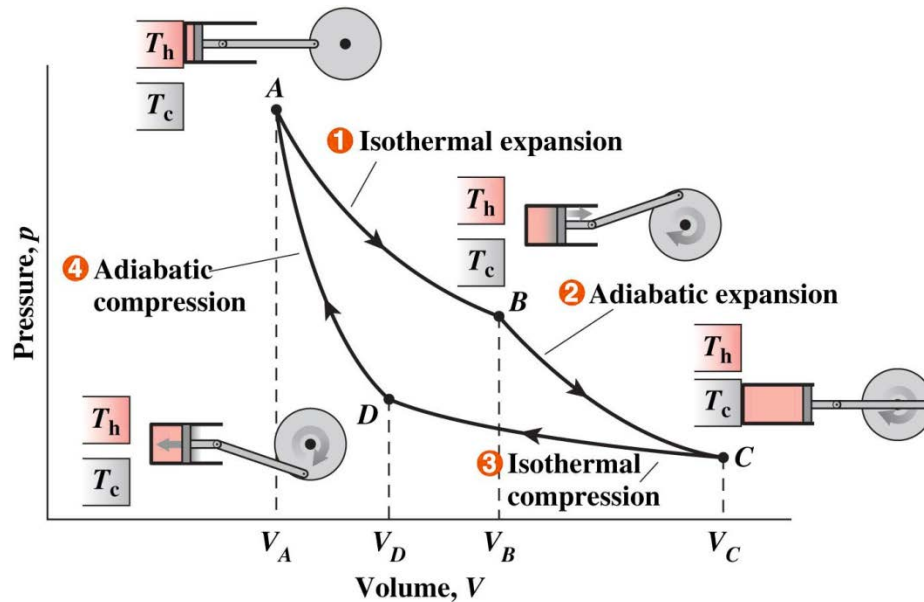
Clausius Statement of the Second Law of Thermodynamics

It is impossible to construct a cyclic refrigerator whose sole effect is the transfer of energy from a cold reservoir to a warm reservoir without the input of energy by work.

Lecture 7. Thermodynamics

Carnot Engine

The Carnot Cycle is the archetypical reversible cycle, and a Carnot Cycle-based heat engine is one that does not dissipate any energy internally and uses only reversible steps. Carnot's Theorem states that no real heat engine operating between a hot reservoir at temperature T_H and a cold reservoir at temperature T_C can be more efficient than a Carnot engine operating between those two reservoirs.



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<http://www.physics.louisville.edu>

A Carnot Cycle consists of four steps:

- Isothermal expansion (in contact with the heat reservoir)
- Adiabatic expansion (after the heat reservoir is removed)
- Isothermal compression (in contact with the cold reservoir)
- Adiabatic compression (after the cold reservoir is removed)

The efficiency of a Carnot Engine is:

$$\epsilon_{\text{Carnot}} = 1 - \frac{T_C}{T_H}$$



Lecture 7. Thermodynamics

Entropy

Entropy S is a measure of disorder. The change in entropy of a system can be evaluated by integrating:

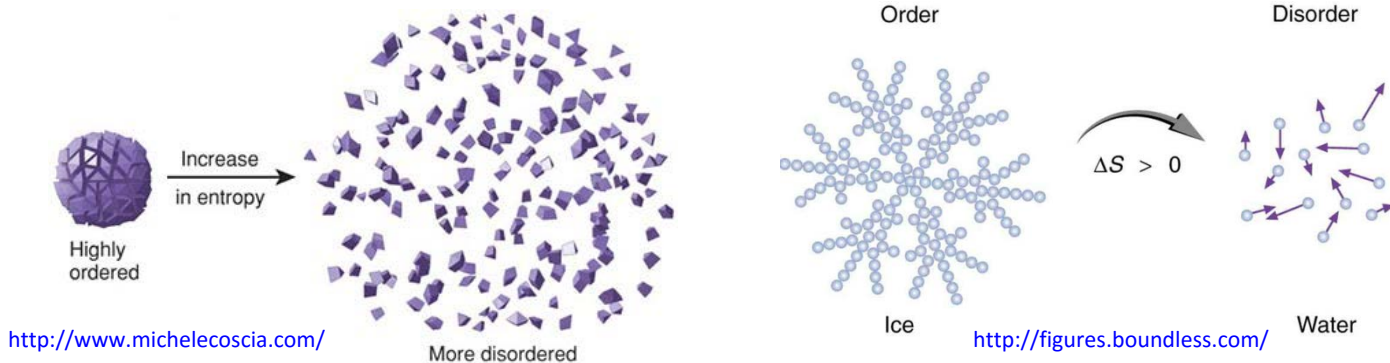
$$dS = \frac{dQ}{T}$$

between successive infinitesimally separated equilibrium states (the weasel language is necessary because temperature should be constant in equilibrium, but systems in equilibrium have constant entropy). Thus:

$$\Delta S = \int \frac{dQ}{T}$$

Entropy Statement of the Second Law of Thermodynamics:

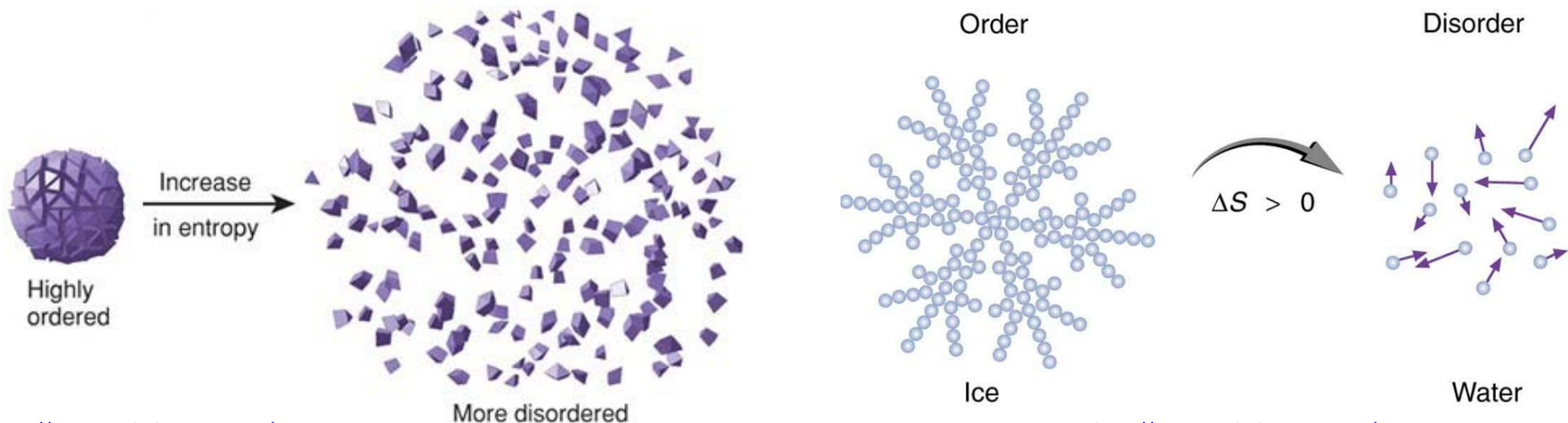
The entropy of the Universe never decreases. It either increases (for irreversible processes) or remains the same (for reversible processes).



Lecture 7. Thermodynamics

Entropy Statement of the Second Law of Thermodynamics

The entropy of the Universe never decreases. It either increases (for irreversible processes) or remains the same (for reversible processes).



Lecture 7. Thermodynamics

Phases and Phase Transitions

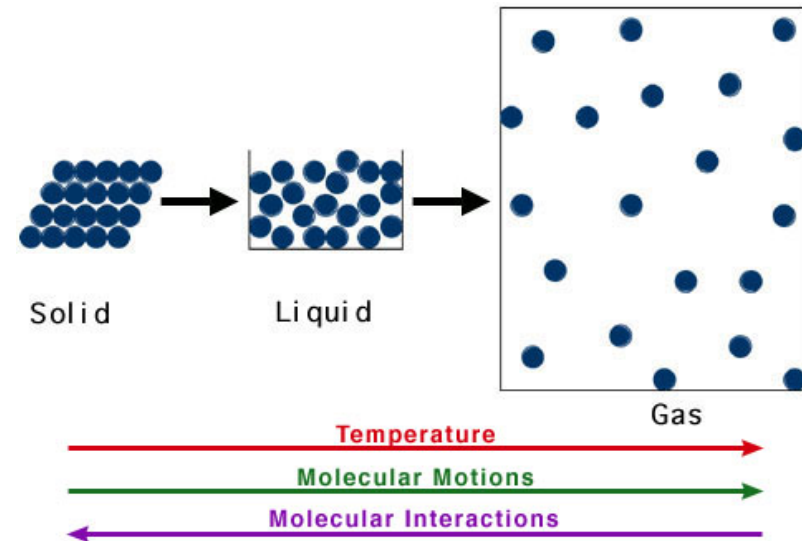
Matter can exist in three different *phases* (*physical states*):

- Solid
- Liquid
- Gas

A phase is a form of matter that is uniform throughout in chemical composition and physical properties, and that can be distinguished from other phases with which it may be in contact by these definite properties and composition.

As shown in Figure:

- a substance in the solid phase has a definite shape and rigidity;
- a substance in the liquid phase has no definite shape, but has a definite volume;
- a substance in the gas phase has no definite shape or volume, but has a shape and volume determined by the shape and size of the container.





Lecture 7. Thermodynamics

Phases and Phase Transitions

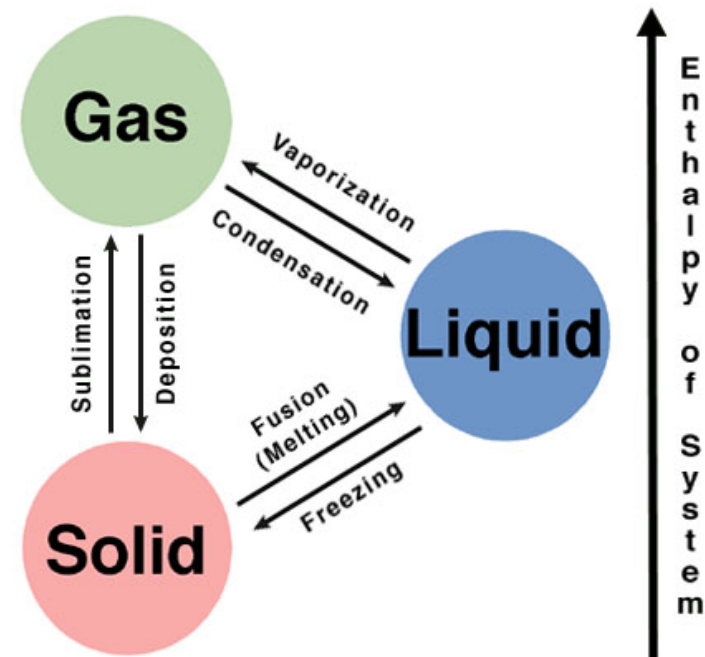
One of the major differences in the three phases is the number of intermolecular interactions they contain.

The particles in a solid interact with all of their nearest neighbors

The particles in a liquid interact with only some of the nearby particles

The particles in a gas ideally have no interaction with one another.

By breaking or forming intermolecular interactions, a substance can change from one phase to another.



For example, gas molecules condense to form liquids because of the presence of attractive intermolecular forces. The stronger the attractive forces, the greater the stability of the liquid (which leads to a higher boiling point temperature). A transition between the phases of matter is called a ***phase transition***. The names of the phase transitions between solid, liquid, and gas are shown in Figure



Lecture 7. Thermodynamics

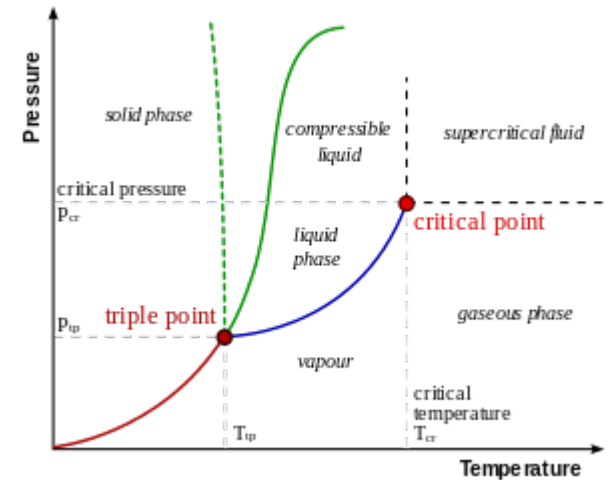
Phases and Phase Transitions

Phase transitions involving *the breaking of intermolecular attractions* (i.e., fusion (melting), vaporization, and sublimation) require an *input* of energy to overcome the attractive forces between the particles of the substance.

Phase transitions involving *the formation of intermolecular attractions* (i.e., freezing, condensation, and deposition) *release energy* as the particles adopt a lower-energy conformation.

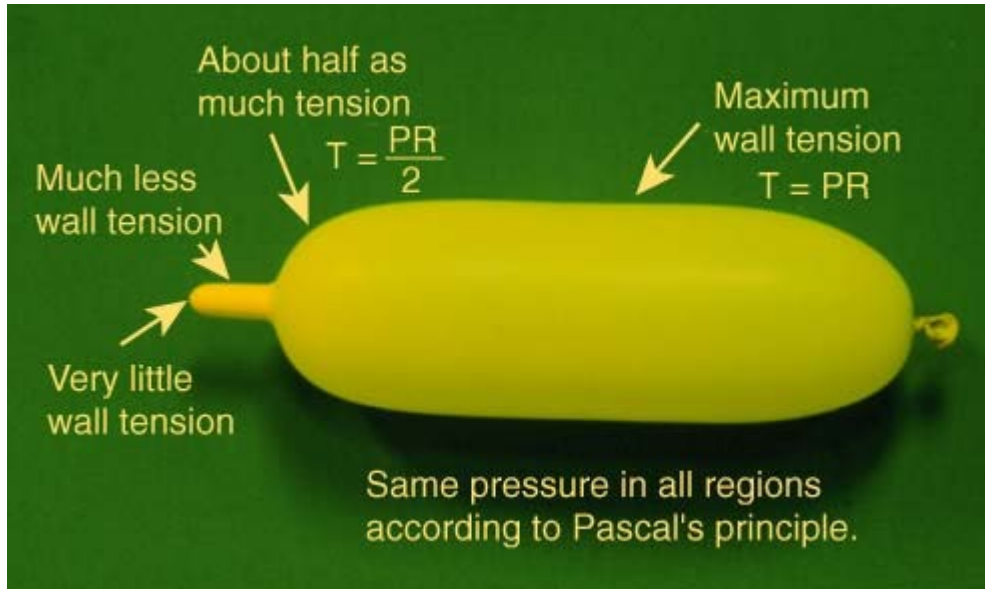
The strength of the intermolecular attractions between molecules, and therefore the amount of energy required to overcome these attractive forces (as well as the amount of energy released when the attractions are formed) depends on the molecular properties of the substance.

In thermodynamics, *the triple point* of a substance is the temperature and pressure at which the three phases (gas, liquid, and solid) of that substance coexist in thermodynamic equilibrium.



Lecture 7. Thermodynamics

Surface Tension and Bubbles

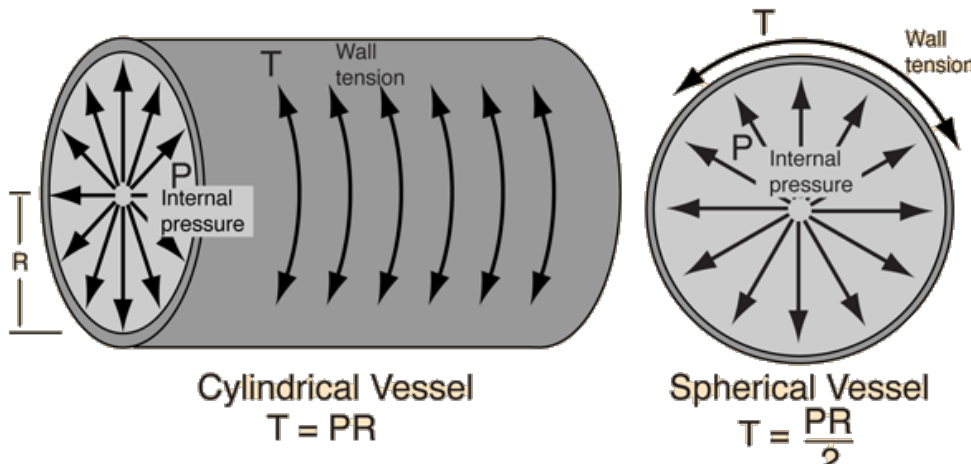


Pascal's principle requires that the pressure is everywhere the same inside the balloon at equilibrium.

But examination immediately reveals that there are great differences in wall tension on different parts of the balloon.

The variation is described by **Laplace's Law**:

The larger the vessel radius, the larger the wall tension required to withstand a given internal fluid pressure.

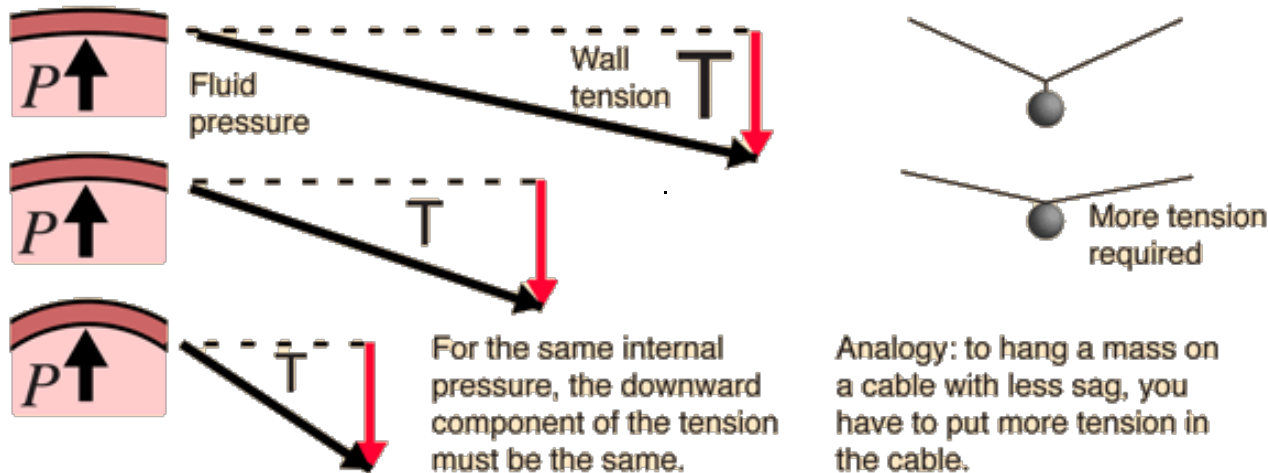


The larger the vessel radius, the larger the wall tension required to withstand a given internal fluid pressure.

Lecture 7. Thermodynamics

Surface Tension and Bubbles

Why does wall tension increase with radius?



If the upward part of the fluid pressure remains the same, then the downward component of the wall tension must remain the same. But if the curvature is less, then the total tension must be greater in order to get that same downward component of tension.

For equilibrium of a load hanging on a cable, you can explore the effects of having a smaller angle for the supporting cable tension.



Lecture 7. Thermodynamics

Surface Tension and Bubbles

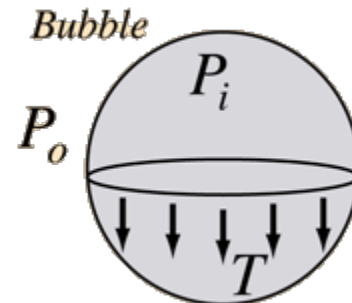
The surface tension of water provides the necessary wall tension for the formation of bubbles with water. The tendency to minimize that wall tension pulls the bubbles into spherical shapes (LaPlace's law).

The interference colors indicate that the thickness of the soap film is on the order of a few wavelengths of visible light. Even though the soap film has less surface tension than pure water, which would pull itself into tiny droplets, it is nevertheless strong to be able to maintain the bubble with such a small thickness.

The pressure difference between the inside and outside of a bubble depends upon the surface tension and the radius of the bubble. The relationship can be obtained by visualizing the bubble as two hemispheres and noting that the internal pressure which tends to push the hemispheres apart is counteracted by the surface tension acting around the circumference of the circle.

For a bubble with two surfaces providing tension, the pressure relationship is:

$$P_i - P_o = \frac{4T}{r}$$





Lecture 7. Thermodynamics

Surface Tension and Bubbles

Surface tension is responsible for the shape of liquid droplets. Although easily deformed, droplets of water tend to be pulled into a spherical shape by the cohesive forces of the surface layer. The spherical shape minimizes then necessary "wall tension" of the surface layer according to LaPlace's law.

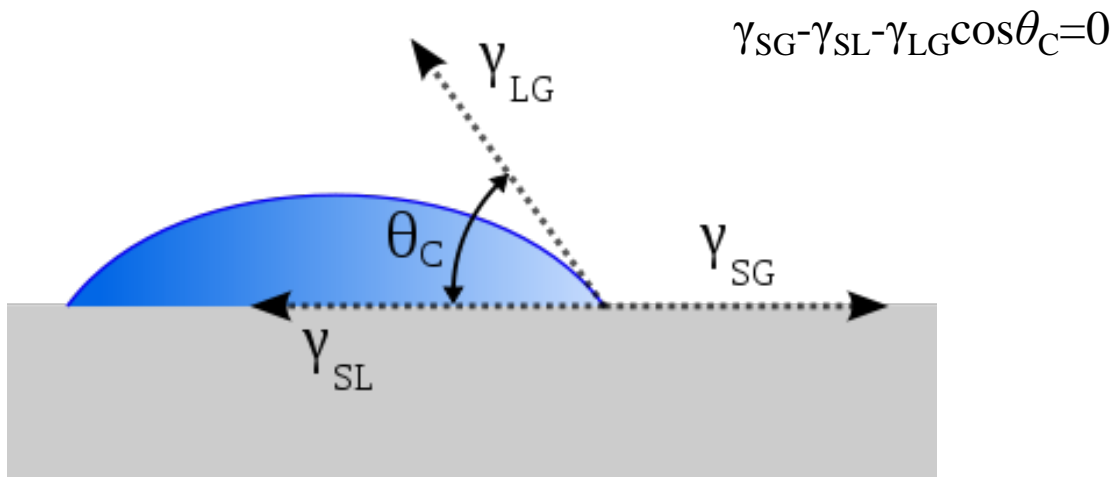


Surface tension and adhesion determine the shape of this drop on a twig. It dropped a short time later, and took a more nearly spherical shape as it fell. Falling drops take a variety of shapes due to oscillation and the effects of air friction.

Lecture 7. Thermodynamics

Surface Tension and Bubbles

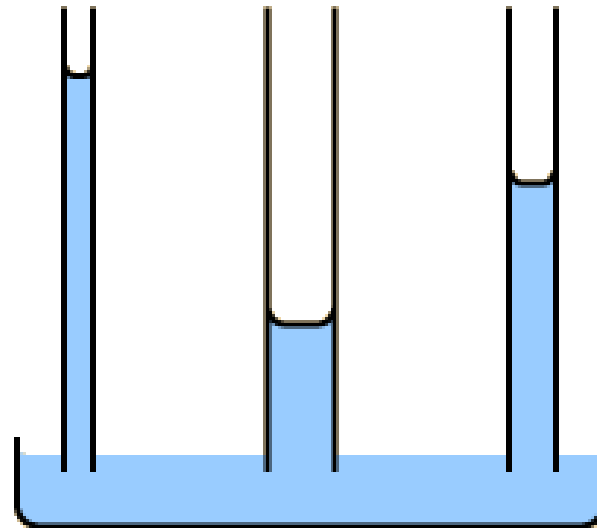
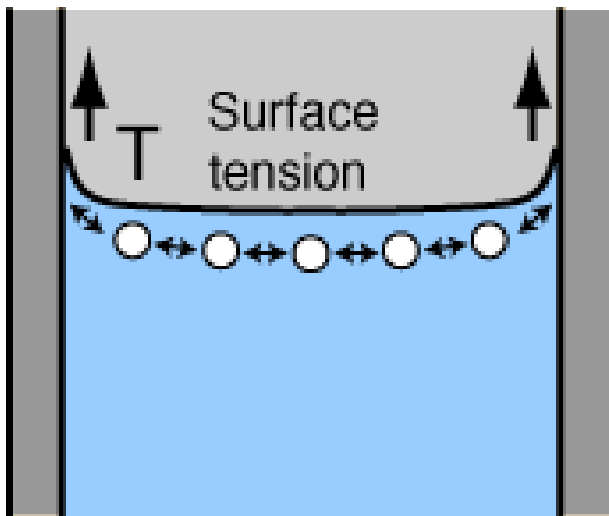
The theoretical description of contact arises from the consideration of a thermodynamic equilibrium between the three phases: the liquid phase (L), the solid phase (S), and the gas/vapor phase (G) (which could be a mixture of ambient atmosphere and an equilibrium concentration of the liquid vapor). The “gaseous” phase could also be another (immiscible) liquid phase. If the solid–vapor interfacial energy is denoted by γ_{SG} , the solid–liquid interfacial energy by γ_{SL} , and the liquid–vapor interfacial energy (i.e. the surface tension) by γ_{LG} , then the equilibrium contact angle θ_C is determined from these quantities by Young's Equation:



Lecture 7. Thermodynamics

Capillary Action

Capillary action is the result of adhesion and surface tension. Adhesion of water to the walls of a vessel will cause an upward force on the liquid at the edges and result in a meniscus which turns upward. The surface tension acts to hold the surface intact, so instead of just the edges moving upward, the whole liquid surface is dragged upward.





Lecture 7. Thermodynamics

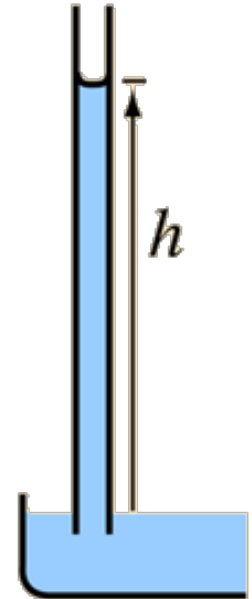
Capillary Action

Capillary action occurs when the adhesion to the walls is stronger than the cohesive forces between the liquid molecules. The height to which capillary action will take water in a uniform circular tube is limited by surface tension.

The height h to which capillary action will lift water depends upon the weight of water which the surface tension will lift:

$$T2\pi r = \rho g(h\pi r^2)$$

The height to which the liquid can be lifted is given by $h = \frac{2T}{\rho r g}$



Since it is weight limited, it will rise higher in a smaller tube.



Lecture 8. Gravity

Early Astronomy

- As far as we know, humans have always been interested in the motions of objects in the sky.
- Not only did early humans navigate by means of the sky, but the motions of objects in the sky predicted the changing of the seasons, etc.





Lecture 8. Gravity

Early Astronomy

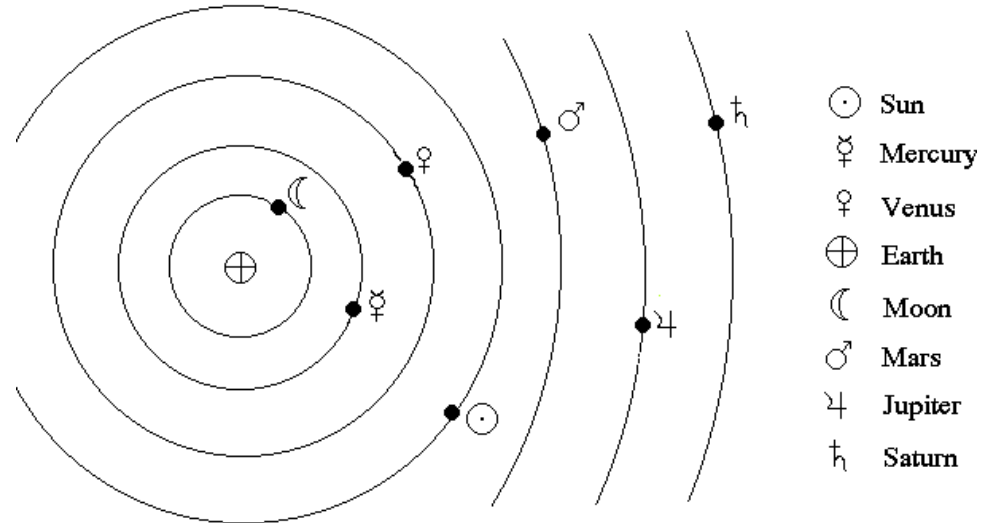
- There were many early attempts both to describe and explain the motions of stars and planets in the sky.
- All were unsatisfactory, for one reason or another.

Lecture 8. Gravity

The Earth-Centered Universe

- A **geocentric** (Earth-centered) solar system is often credited to **Ptolemy**, an Alexandrian Greek, although the idea is very old.

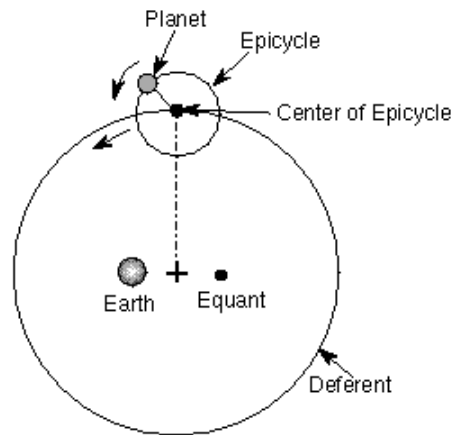
Geocentric Solar System Model



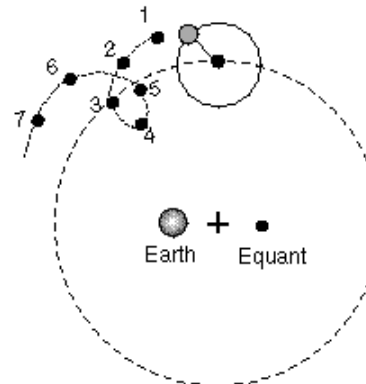
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Ptolemy's Solar System

- Ptolemy's solar system could be made to fit the observational data pretty well, but only by becoming *very* complicated.



Center of epicycle moves counter-clockwise on deferent and epicycle moves counterclockwise. Epicycle speed is uniform with respect to equant. The combined motion is shown at right.

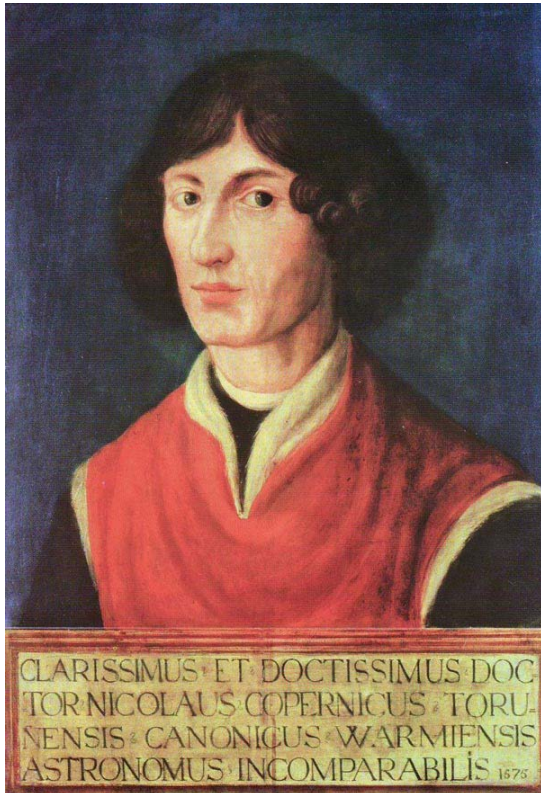


Deferent motion is in direction of point 1 to 7 but planet's epicycle carries it on cycloid path (points 1 through 7) so that from points 3 through 5 the planet moves backward (retrograde).

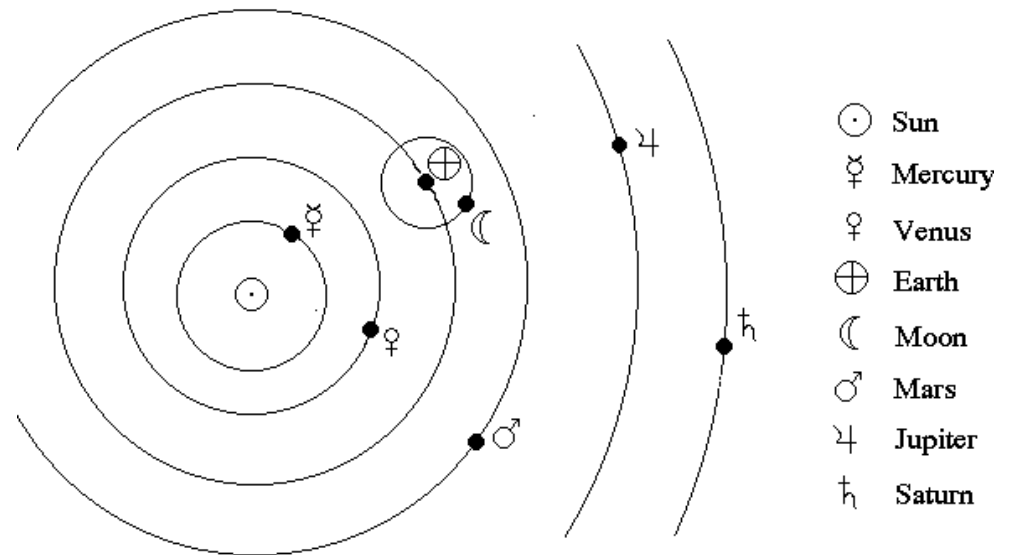
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Copernicus' Solar System

- The Polish cleric **Copernicus** proposed a **heliocentric** (Sun centered) solar system in the 1500's.



Heliocentric Solar System Model





Lecture 8. Gravity

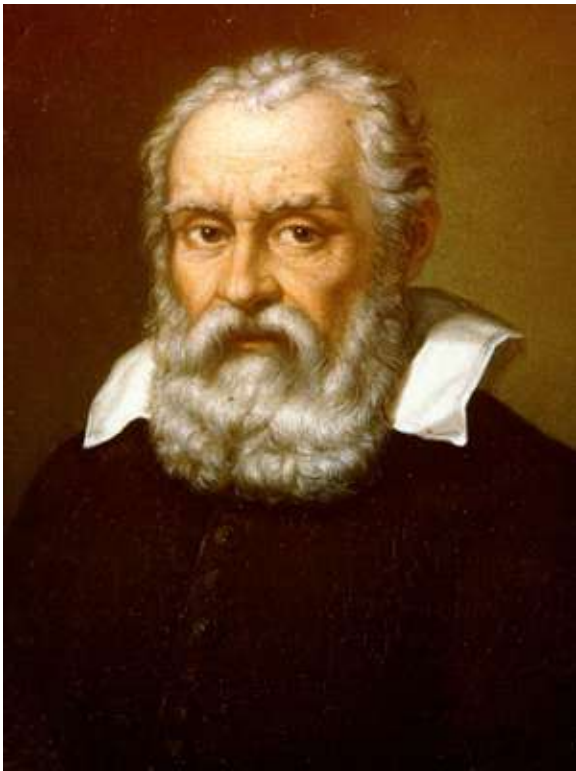
Objections to Copernicus

- How could Earth be moving at enormous speeds when we don't feel it?
 - (Copernicus didn't know about *inertia*.)
- Why can't we detect Earth's motion against the background stars (stellar parallax)?
- Copernicus' model did **not** fit the observational data very well.



Lecture 8. Gravity

Galileo & Copernicus



- Galileo became convinced that Copernicus was correct by observations of the Sun, Venus, and the moons of Jupiter using the newly-invented telescope.
- Perhaps Galileo was motivated to understand inertia by his desire to understand and defend Copernicus' ideas.



Lecture 8. Gravity

Tycho and Kepler



- In the late 1500's, a Danish nobleman named *Tycho Brahe* set out to make the *most accurate measurements* of planetary motions to date, in order to validate his own ideas of planetary motion.



Lecture 8. Gravity

Tycho and Kepler



- Tycho's data was successfully interpreted by the German mathematician and scientist *Johannes Kepler* in the early 1600's.



Lecture 8. Gravity

Kepler's Laws

The laws themselves are surprisingly simple and geometric:

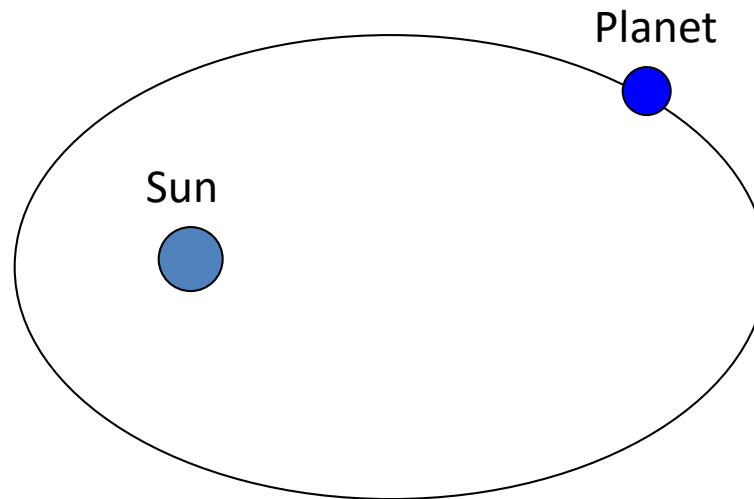
- a) Planets move around the Sun in elliptical orbits with the Sun at one focus.
- b) Planets sweep out equal areas in equal times as they orbit the Sun.
- c) The mean radius of a planetary orbit (in particular, the semimajor axis of the ellipse) cubed is directly proportional to the period of the planetary orbit squared, with the same constant of proportionality for all of the planets.



Lecture 8. Gravity

Early Astronomy

- **Kepler** determined that the orbits of the planets were not perfect circles, but *ellipses*, with the Sun at one focus.

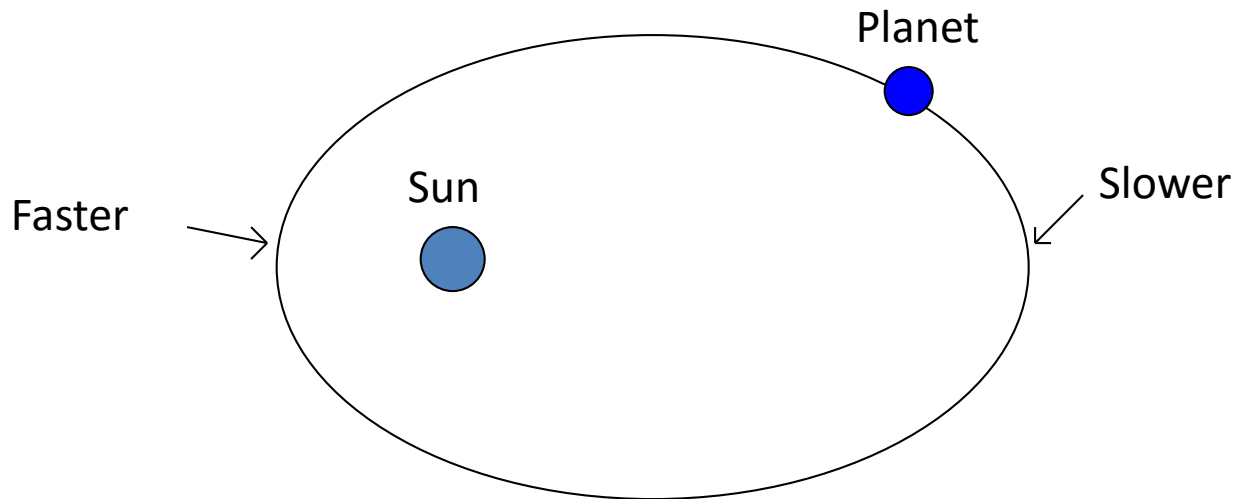




Lecture 8. Gravity

Kepler's Second Law

- Kepler determined that a planet moves **faster** when **near** the Sun, and **slower** when **far** from the Sun.





Lecture 8. Gravity

Kepler's Laws

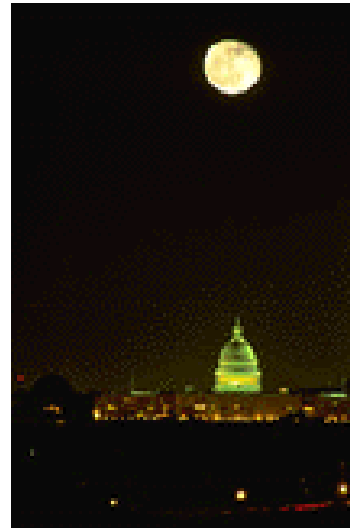
- Kepler's Laws provided a complete *kinematical* description of planetary motion (including the motion of planetary satellites, like the Moon) - but *why* did the planets move like that?



Lecture 8. Gravity

The Apple & the Moon

- Isaac Newton realized that the motion of a falling apple and the motion of the Moon were both actually the *same motion*, caused by the *same force* - the *gravitational force*.

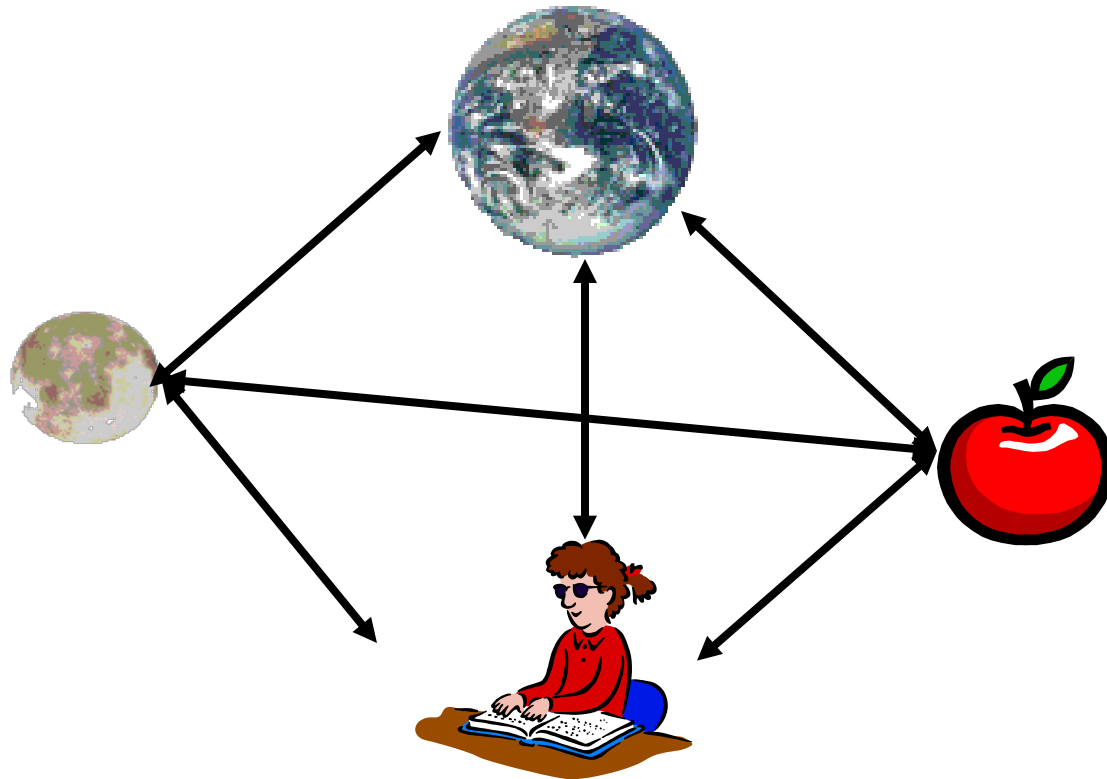




Lecture 8. Gravity

Universal Gravitation

- Newton's idea was that gravity was a *universal* force acting between *any two objects*.

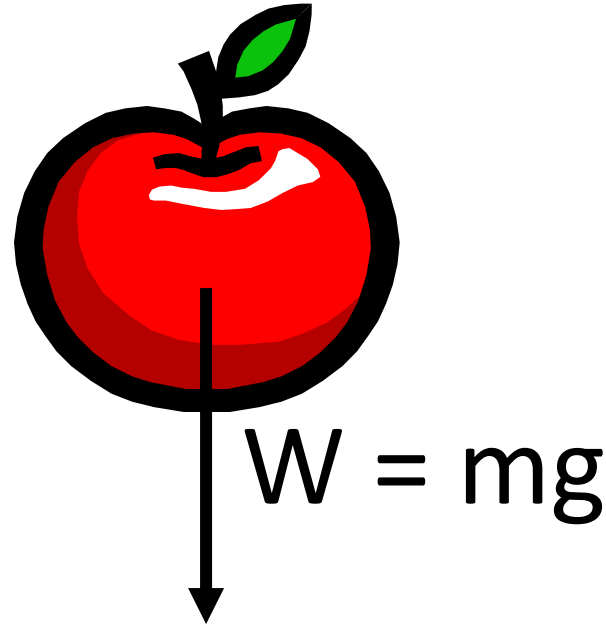




Lecture 8. Gravity

At the Earth's Surface

- Newton knew that the *gravitational force* on the apple equals the apple's *weight, mg* , where $g = 9.8 \text{ m/s}^2$.

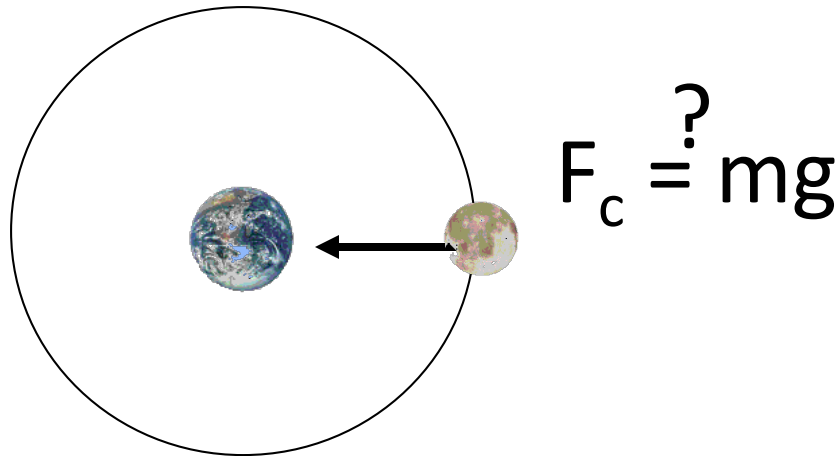




Lecture 8. Gravity

Weight of the Moon

- Newton reasoned that the centripetal force on the moon was also supplied by the Earth's gravitational force.

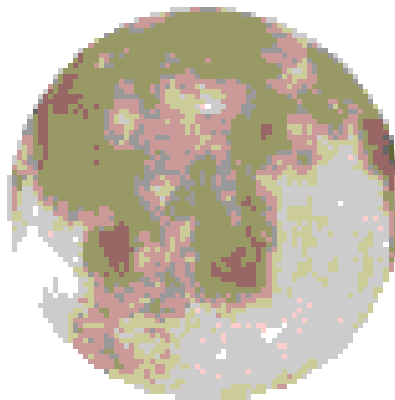




Lecture 8. Gravity

Weight of the Moon

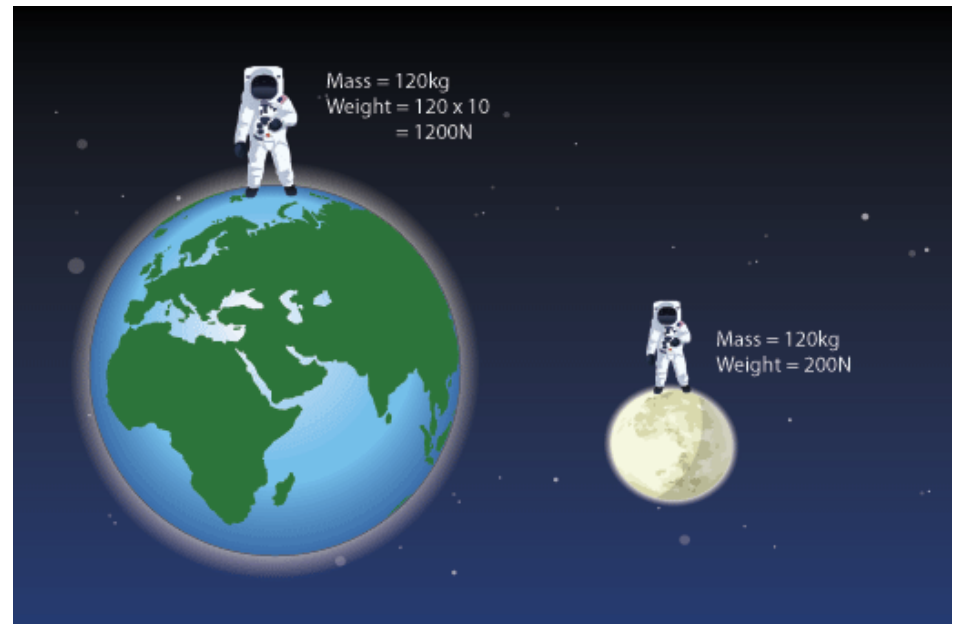
- Newton's calculations showed that the centripetal force needed for the Moon's motion was about $1/3600^{\text{th}}$ of Mg , however, where M is the mass of the Moon.



Lecture 8. Gravity

Weight of the Moon

- Newton knew, though, that the Moon was about **60 times farther** from the center of the Earth than the apple.
- And $60^2 = 3600$





Lecture 8. Gravity

Universal Gravitation

- From this, Newton reasoned that the strength of the gravitational force is *not constant*, in fact, the magnitude of the force is *inversely proportional to the square of the distance* between the objects.
- Newton concluded that the gravitational force is:
 - *Directly proportional* to the *masses* of *both* objects.
 - *Inversely proportional* to the *distance* between the objects.

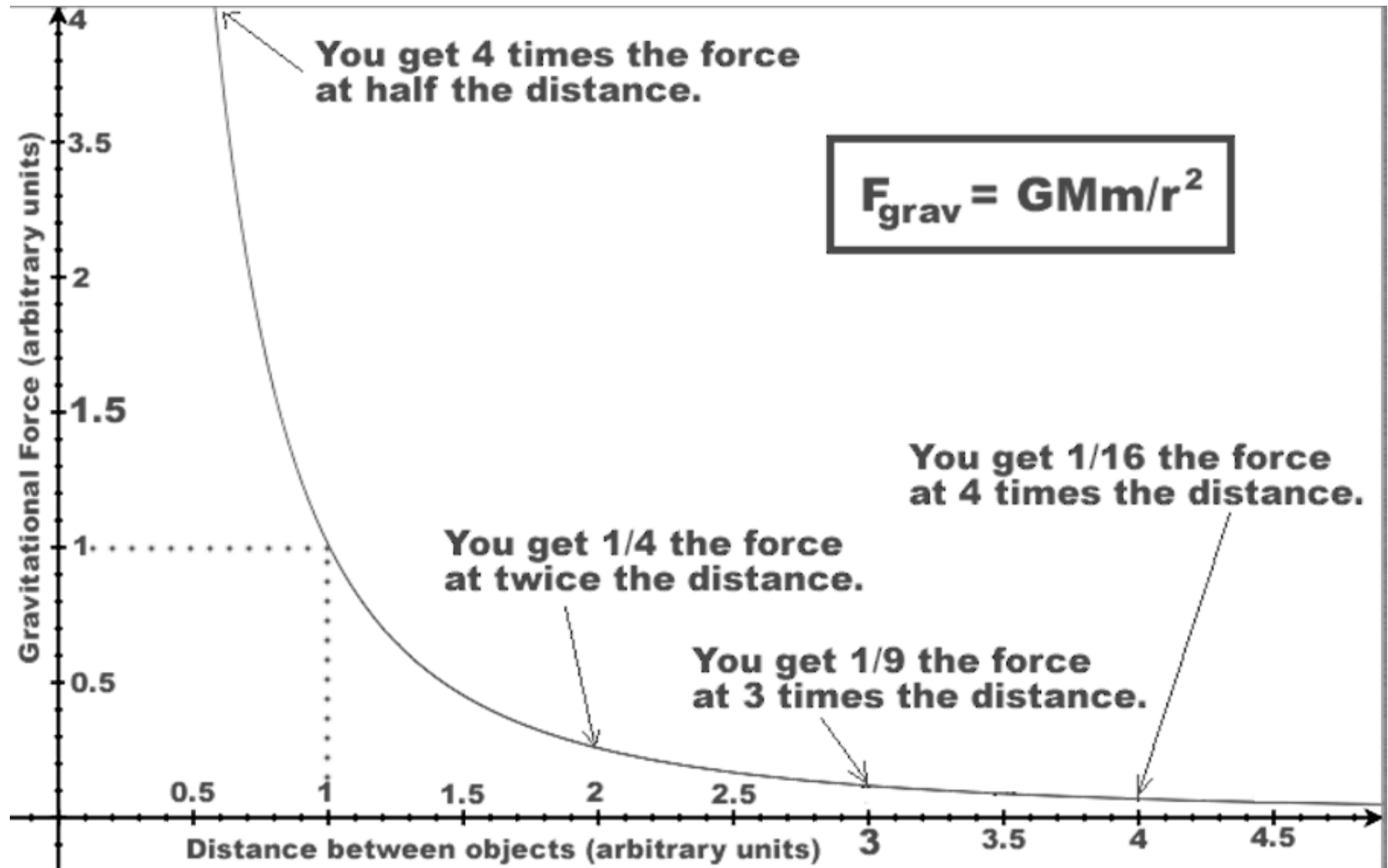
$$\vec{F}_{21} = -\frac{GM_1m_2}{r^2}\hat{r}$$

where $G = 6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2$ is the universal gravitational constant

- Newton's Law of Universal Gravitation is often called an *inverse square law*, since the force is inversely proportional to the square of the distance.



Lecture 8. Gravity





Lecture 8. Gravity

Experimental Evidence

- The Law of Universal Gravitation allowed **extremely accurate predictions** of planetary orbits.
- **Cavendish** measured gravitational **forces between human-scale objects** before 1800. His experiments were later simplified and improved by von Jolly.
- In Newton's time, there was much discussion about HOW gravity worked - how does the Sun, for instance, reach across empty space, with no actual contact at all, to exert a force on the Earth?
- This spooky notion was called "**action at a distance.**"



Lecture 8. Gravity

The Gravitational Field

- During the 19th century, the notion of the “*field*” entered physics (via Michael Faraday).
- Objects with mass create an *invisible disturbance in the space around them* that is felt by other massive objects - this is a *gravitational field*.
- So, since the Sun is very massive, it creates an intense gravitational field around it, and the *Earth responds to the field*. No more “action at a distance.”



Lecture 8. Gravity

Gravitational Field Strength

- To measure the strength of the gravitational field at any point, measure the gravitational force, F , exerted on any “test mass”, m .
- *Gravitational Field Strength, $g = F/m$*
- Near the surface of the Earth, $g = F/m = 9.8 \text{ N/kg} = 9.8 \text{ m/s}^2$.
- In general, $g = GM/r^2$, where M is the mass of the object creating the field, r is the distance from the object's center, and $G = 6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2$.



Lecture 8. Gravity

Gravitational Force

- If g is the strength of the gravitational field at some point, then the gravitational force on an object of mass m at that point is $F_{grav} = mg$.
- If g is the gravitational field strength at some point (in N/kg), then the free fall acceleration at that point is also g (in m/s²).

Gravitational Field Inside a Planet

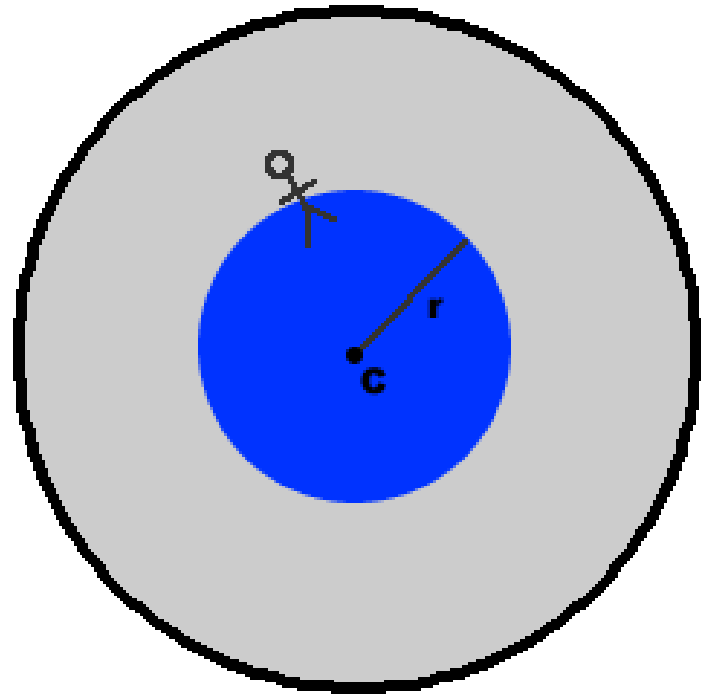
- If you are located a distance r from the center of a planet:
 - all of the planet's mass **inside** a sphere of radius r *pulls you toward the center of the planet*.
 - All of the planet's mass **outside** a sphere of radius r exerts **no** net **gravitational force** on you.



Lecture 8. Gravity

Gravitational Field Inside a Planet

- The blue-shaded part of the planet pulls you toward point C.
- The grey-shaded part of the planet does not pull you at all.

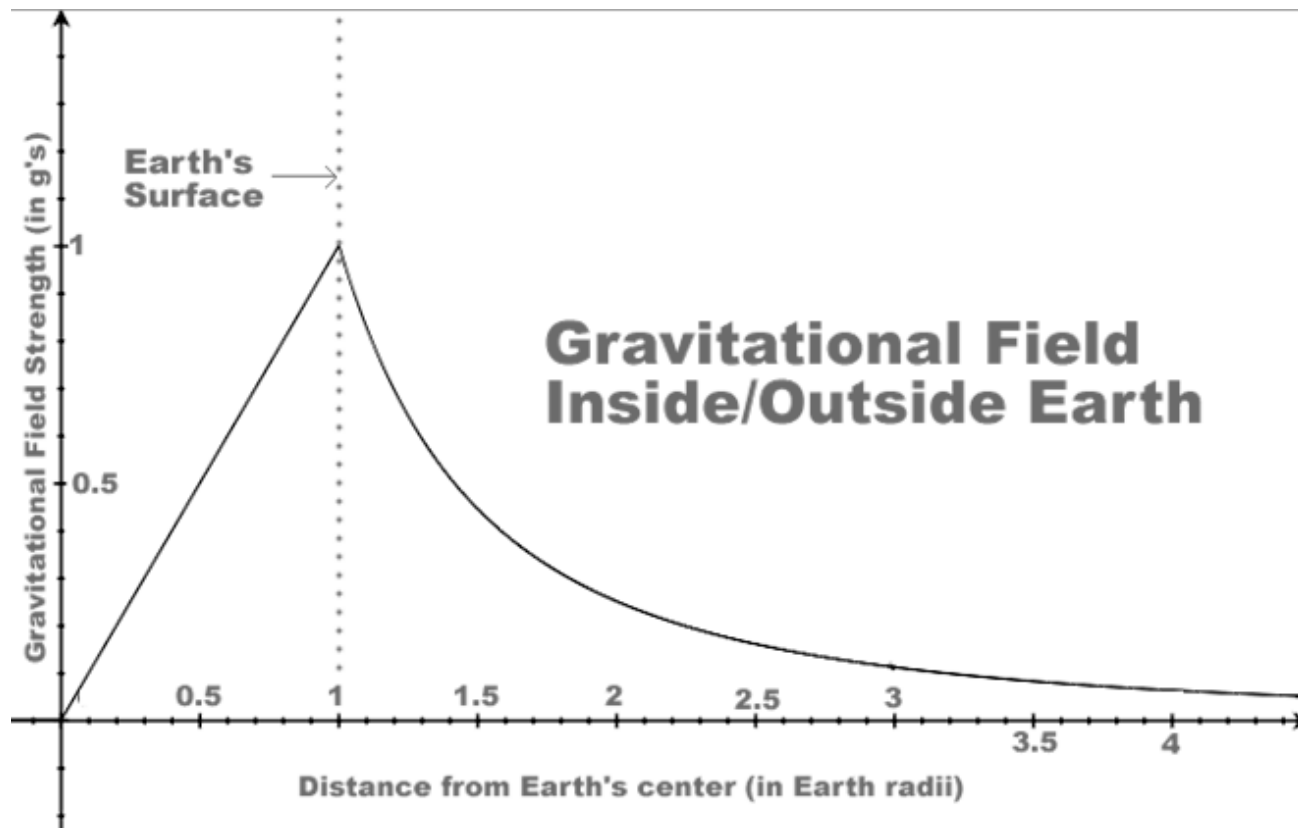




Lecture 8. Gravity

Gravitational Field Inside a Planet

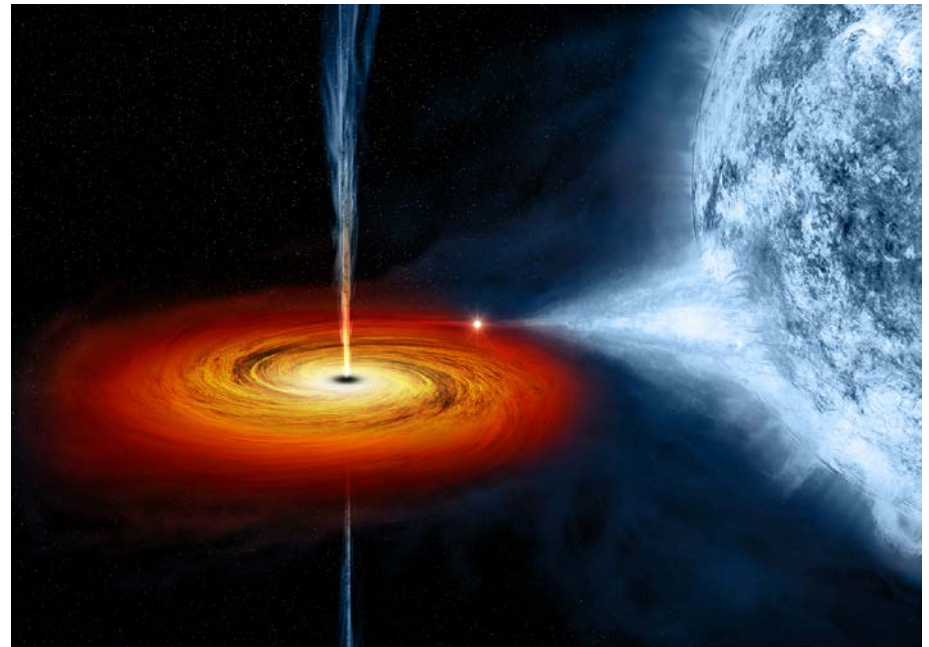
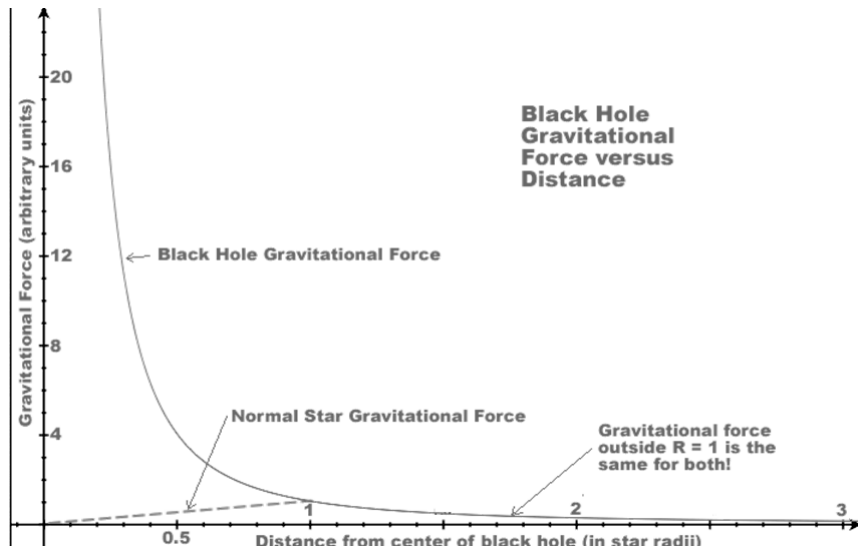
- Half way to the center of the planet, g has one-half of its surface value.
- At the center of the planet, $g = 0 \text{ N/kg}$.



Lecture 8. Gravity

Black Holes

- When a very massive star gets old and runs out of fusionable material, gravitational forces may cause it to collapse to a mathematical point - a **singularity**. All normal matter is crushed out of existence. This is a **black hole**.





Lecture 8. Gravity

Black Hole Gravitational Force

- The black hole's gravity is the **same as the original star's** at distances greater than the star's original radius.
- Black hole's **don't** magically “suck things in.”
- The black hole's gravity is intense because you can get really, really close to it!

Earth's Tides

- There are 2 high tides and 2 low tides per day.
- The tides follow the Moon.



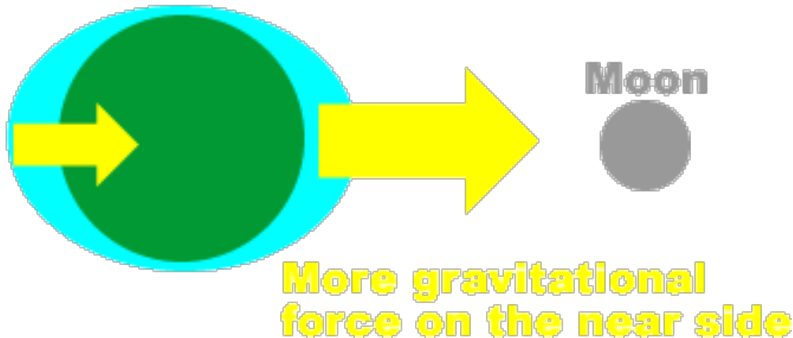


Lecture 8. Gravity

Why Two Tides?

- Tides are caused by the **stretching** of a planet.
- Stretching is caused by a **difference in forces** on the two sides of an object.
- Since **gravitational force depends on distance**, there is more gravitational force on the side of Earth closest to the Moon and less gravitational force on the side of Earth farther from the Moon.

**Less gravitational force
on the far side.**



$$F_{Grav} = G \frac{Mm}{r^2}$$



Lecture 8. Gravity

Why the Moon?

- The Sun's gravitational pull on Earth is **much larger** than the Moon's gravitational pull on Earth. So why do the tides follow the Moon and not the Sun?
- Since the Sun is much farther from Earth than the Moon, the difference in distance across Earth is much less significant for the Sun than the Moon, therefore the difference in gravitational force on the two sides of Earth is less for the Sun than for the Moon (even though the Sun's force on Earth is more).
- The Sun does have a small effect on Earth's tides, but the major effect is due to the Moon.

