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Homogeneous Differential-Operator Equations in Locally Convex Spaces

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Abstract—We describe a general method that allows us to find solutions to homogeneous differential-operator equations with variable coefficients by means of continuous vector-valued functions. The "homogeneity" is not interpreted as the triviality of the right-hand side of an equation. It is understood in the sense that the left-hand side of an equation is a homogeneous function with respect to operators appearing in that equation. Solutions are represented as functional vector-valued series which are uniformly convergent and generated by solutions to a kth order ordinary differential equation, by the roots of the characteristic polynomial and by elements of a locally convex space. We find sufficient conditions for the continuous dependence of the solution on a generating set. We also solve the Cauchy problem for the considered equations and specify conditions for the existence and the uniqueness of the solution. Moreover, under certain hypotheses we find the general solution to the considered equations. It is a function which yields any particular solution. The investigation is realized by means of characteristics of operators such as the order and the type of an operator, as well as operator characteristics of vectors, namely, the operator order and the operator type of a vector relative to an operator. We also use a convergence of operator series with respect to an equicontinuous bornology.

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INTRODUCTION

Let $\mathbf{H}$ be a separated quasi-complete locally convex space over the field of complex numbers with a multinorm $\mathbb{P} = \{\|\cdot\|_p\}_{p\in\mathcal{P}}$. Let $\mathcal{H}_G$ be the space of all vector-functions taking the values in the space $\mathbf{H}$ which are continuous on a set $G \subset \mathbb{R}$ or analytic in a domain $G \subset \mathbb{C}$. This space is endowed with the topology of uniform convergence on compact sets:

$$\|u\|'_{p,r} = \max_{t\in G_r} \|u(t)\|_p, \;\; G_r \subset G, \;\; \bigcup_r G_r = G, \;\; u \in \mathcal{H}_G.$$

If a vector-function u belongs to the space $\mathcal{H}_G$, then $u(t)$ is a vector in $\mathbf{H}$ for each fixed point $t \in G$. For an operator A, setting $v(t) = A(u(t))$, we obtain a vector-function $v \in \mathcal{H}_G$. In this way, one defines the operator $\mathcal{A} : \mathcal{H}_G \to \mathcal{H}_G$. The continuity of the operator A implies the continuity of the operator $\mathcal{A}$.

Let us consider a differential-operator equation

$$\sum_{j=0}^{m} a_{m-j}\mathcal{A}^j\mathcal{D}^{m-j}u = f, \;\; a_i \in \mathbb{C}, \;\; a_m = 1, \;\; f \in \mathcal{H}_G, \tag{1}$$

where $\mathcal{D}$ is a kth order linear differential operator, in general, with variable coefficients, acting on the space of vector-functions $\mathcal{H}_G$, and u is an unknown vector-function.

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