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ON THE RADIUS OF n-VALENT CONVEXITY AND STARLIKENESS FOR SOME SPECIAL CLASSES OF ANALYTICAL FUNCTIONS ON A DISK

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Let us denote by $\tau^n_{\alpha,\theta}$, $0 \le \alpha < 1$, $0 \le \theta \le \pi$, $n = 1, 2, 3, \dots$, the class of functions $w = g(z)$; $g(0) = 0$, $g'(0) = 0$, $g''(0) = 0, \dots, g^{(n-1)}(0) = 0$, $g^{(n)}(0) = (n-1)!$, which are regular in the disk $E = \{z : |z| < 1\}$ and satisfy the following condition

$$\operatorname{Re}\left(\frac{1 - 2z^n \cos\theta + z^{2n}}{z^{n-1}} g'(z) \right) > \alpha. \tag{1}$$

Since

$$\varphi_{n,\theta} = \int_0^z (1 - 2\zeta^n \cos\theta + \zeta^{2n})^{-1} \zeta^{n-1} d\zeta$$

belongs to the class of convex functions, the class $\tau^1_{\alpha,\theta}$ is a subset of the class of almost convex functions (see [1]). Consequently, it consists of univalent functions.

For $n = 1$, the function $\psi(z) = \frac{1}{i} g(iz)$ satisfies the inequality

$$\operatorname{Re}[(1 - 2iz\cos\theta - z^2)\psi'(z)] > \alpha.$$

For $\theta = \pi/2$, we obtain here a subset Q_α (see [2]) of the class of almost convex functions $w = \psi(z)$, $\psi(0) = 0$, $\psi'(0) = 1$, defined via the condition

$$\operatorname{Re}[(1 - z^2)\psi'(z)] > \alpha. \tag{2}$$

Note that for $\alpha = 0$ relation (2) defines a class of functions which are convex in the direction of the imaginary axis.

From (1) we obtain that

$$g'(z) = z^{n-1} \frac{p(z) + h}{(1+h)(1 - 2z^n \cos\theta + z^{2n})}, \quad h = \frac{\alpha}{1-\alpha},$$

where $p(z)$ belongs to the class P of functions which are regular in E and satisfy in E the conditions $p(0) = 1$, $\operatorname{Re} p(z) > 0$.

Let us denote by $O^n_{\alpha,\theta}$ the class of functions regular in E: $w = s(z)$, $s(0) = s'(0) = s''(0) = \cdots = s^{(n-1)}(0) = 0$, $s^{(n)}(0) = n!$, such that $s(z) = zg'(z)$, $g(z) \in \tau^n_{\alpha,\theta}$.

Let us establish the exact convexity bounds in the class $\tau^n_{\alpha,\theta}$. For the functions of this class we have

$$\operatorname{Re}\left(1 + z\frac{g''(z)}{g'(z)}\right) = \operatorname{Re}\left(\frac{zp'(z)}{p(z)+h} + n\frac{1 - z^{2n}}{1 - 2z^n\cos\theta + z^{2n}} \right). \tag{3}$$