ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2011, Vol. 55, No. 10, pp. 70–77. © Allerton Press, Inc., 2011.
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A Complete Description of the Lebesgue Functions for Classical Lagrange Interpolation Polynomials

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Received August 24, 2010

Abstract—Explicit forms of Lebesgue functions are not described in the mathematical literature by now. This issue is related to the problem of reducing sums of modules of fundamental polynomials or the corresponding Dirichlet kernels. That is why the complete study of graphs of Lebesgue functions remains a complicated topical problem in the theory of approximation. In this paper we solve the mentioned problems both for odd and even numbers of interpolation nodes. We find explicit forms of Lebesgue functions and study them by means of the differential calculus. All mentioned forms are new.

DOI: 10.3103/S1066369X11100094

Keywords and phrases: *Lebesgue function, Lebesgue constant, Lagrange interpolation polynomial, Dirichlet kernel.*

As is known (e.g., [1–3]), the convergence of interpolation processes is immediately related to the corresponding Lebesgue functions and constants. We consider the trigonometric interpolation of a continuous 2π-periodic function $x = x(t)$ with N nodes uniformly distributed on $[0; 2\pi]$, i.e.,

$$t_k = \frac{2\pi k}{N} + \frac{2\pi\theta}{N} \quad (\theta \in [0;1];\ \ k = \overline{0, N-1},\ \ N = 2n+1 \vee N = 2n,\ \ n \in \mathrm{N}).$$

Lebesgue functions of this interpolation process are ([1], P. 66; [2], P. 36) finite sums

$$\lambda_n(t) = \frac{2}{2n+1}\sum_{k=0}^{2n} |D_n(t_k - t)| = \frac{2}{2n+1}\sum_{k=1}^{2n+1} |D_n(t_k - t)| \quad (N = 2n+1), \tag{1}$$

$$\lambda_n^*(t) = \frac{1}{n}\sum_{k=0}^{2n-1} |D_n^*(t_k^* - t)| = \frac{1}{n}\sum_{k=1}^{2n} |D_n^*(t_k^* - t)| \quad (N = 2n) \tag{2}$$

which correspond to the well-known Lagrange interpolation polynomials

$$P_n(x;t) = \frac{2}{2n+1}\sum_{k=0}^{2n} x(t_k) D_n(t_k - t) \quad (t_k = 2\pi k/(2n+1)), \tag{3}$$

$$P_n^*(x;t) = \frac{1}{n}\sum_{k=0}^{2n-1} x(t_k^*) D_n^*(t_k^* - t) \qquad (t_k^* = \pi k/n,\ \ n \in \mathrm{N}). \tag{4}$$

In what follows, for definiteness, we assume that either $\theta = 0$ or $\theta = 1$, and the Dirichlet kernels

$$D_n(t) = \frac{0.5\sin(n+0,5)t}{\sin(t/2)}, \quad D_n^*(t) = \frac{0.5\sin nt}{\tan(t/2)}$$

satisfy the conditions

$$D_n(-t) = D_n(t),\ \ D_n(t_0 - t) = D_n(t_{2n+1} - t);\ \ D_n^*(-t) = D_n^*(t),\ \ D_n^*(t_0^* - t) = D_n^*(t_{2n}^* - t). \tag{5}$$

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