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The Hilbert Boundary-Value Problem With a Finite Index and a Countable Set of Jump Discontinuities in Coefficients

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Abstract—We consider the Hilbert boundary-value problem with a finite index for the case, when the coefficients in the boundary condition have two infinite sequences of discontinuity points of the first kind. We obtain a formula for the general solution and study the solvability issues.

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Historically, the first of the basic boundary problems in the theory of analytic functions was the Hilbert problem. The first precise formulation was given in 1883 [1]: Find in a prescribed domain D (in what follows, we assume that D is the upper half-plane) an analytic function $\Phi(\zeta)$ from the boundary condition in the form

$$a(t)\,\mathrm{Re}\,\Phi(t) - b(t)\,\mathrm{Im}\,\Phi(t) = c(t), \tag{1}$$

where coefficients and the right-hand side of the problem, i.e., $a(t)$, $b(t)$, and $c(t)$, are given functions of points on the contour $L = \partial D$. We assume that the following condition is fulfilled:

$$a^2(t) + b^2(t) \neq 0. \tag{2}$$

The total increment of the function $\nu(t) = \arg G(t)$, $G(t) = a(t) - ib(t)$, divided by π, when we go around the boundary of the domain in positive direction, is called *the index of the Hilbert boundary-value problem*. The index of the problem is a characteristic which determines the problem solvability. Let us rewrite the boundary condition in the form

$$\mathrm{Re}\left[e^{-i\nu(t)}\Phi(t)\right] = \frac{c(t)}{|G(t)|}. \tag{3}$$

The full solution to this problem (in the case of a finite index and continuous coefficients) was given by Hilbert. In [2] the investigation of the problem was reduced to studying a singular integral equation, and in [3] the problem was reduced to two Dirichlet problems solved sequentially.

There exist two basic approaches to investigating the Hilbert problem. N. I. Muskhelishvili ([4], pp. 140–145, 302–308) solves the problem by reducing it to the Riemann boundary-value problem. The method proposed by F. D. Gakhov develops the ideas of [3]. It is based on the transformation of the boundary condition in the Hilbert problem to that in a generalized Schwarz problem.

See [4], P. 140–145, and [5], pp. 273–280, for the solution of the Hilbert problem in the case when coefficients and the right-hand side satisfy the Hölder condition, and in a neighborhood of infinity they also satisfy the condition

$$|a(t'') - a(t')| \le K\left|\frac{1}{t''} - \frac{1}{t'}\right|^{\gamma} \tag{4}$$

with constants $K > 0$ and γ, $0 < \gamma \le 1$. A generalization of the problem to the case of piecewise Hölder coefficients with a finite number of discontinuities was investigated in [4], pp. 302–308, [5], pp. 466–468, [6–8]. In [9] one obtains a solution to the Hilbert problem with piecewise Hölder coefficients that

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