*Russian Mathematics (Iz. VUZ)* *Izvestiya VUZ. Matematika*
*Vol. 43, No. 5, pp. 53–60, 1999* *UDC 519.63*

# RESOLVABILITY AND THE FINITE ELEMENT METHOD FOR DEGENERATE ELLIPTIC EQUATIONS OF HIGH ORDER

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The present article is devoted to investigation of existence and uniqueness of generalized solution of a linear degenerate elliptic equation of order $2m$ for two cases. First, for the case where the equation degenerates on a part of the domain's boundary (in particular, on the whole boundary). Second, when the degeneration occurs inside the domain. We analyze the problem's statement in dependence on equation coefficients' degeneration degree by application of the embedding theorem of the weight Sobolev spaces. For approximate solution we suggest the finite element method (FEM).

A series of works are dedicated to both the questions of resolvability of equations and the analysis of the properties of their solutions provided that they degenerate on domain's boundary (see, e. g., [1]–[7] and references therein). Grid methods for second order equations degenerating on a part of the boundary were considered, e. g., in [8]–[17]. The finite element method for a quasilinear 4-th order equation degenerating on a part of the boundary was considered in [18]. In the present article we shall follow the works [15]–[20].

**1.** *Notation and auxiliary results.* Let $\Omega \subset R^n$ be a bounded domain with a Lipschitz-continuous boundary and $S \subset \partial\Omega$ a certain $n-1$-dimensional surface which is sufficiently smooth. We denote by

$$\rho(x) = \inf\{|x-y| : y \in S\}$$

the distance between the point $x$ and the surface $S$. We introduce the weight Lebesgue and Sobolev spaces

$$L_{p,\alpha}(\Omega) = \{u(x) : \rho(x)^\alpha u(x) \in L_p(\Omega)\} \quad (p > 1),$$

$$|u|_{L_{p,\alpha}(\Omega)} = |\rho^\alpha u|_{p,\Omega} = \Big(\int_\Omega |\rho(x)^\alpha u(x)|^p dx\Big)^{1/p};$$

$$W^m_{p,\alpha}(\Omega) = \{u(x) \in L_{1,\mathrm{loc}}(\Omega) : D^i u(x) \in L_{p,\alpha}(\Omega),\ i = (i_1, i_2, \dots, i_n),\ |i| = i_1 + i_2 + \dots + i_n \le m\},$$

$$\|u\|_{W^m_{p,\alpha}(\Omega)} = \Big(\sum_{|i|\le m} \int_\Omega |\rho^\alpha D^i u|^p dx\Big)^{1/p}.$$

For an arbitrary measurable set $K \subset \overline{\Omega}$ we put

$$|\rho^\alpha \nabla^m u|_{p,K} = \Big(\sum_{|i|=m} \int_K |\rho^\alpha D^i u|^p dx\Big)^{1/p}.$$

Supported by Russian Foundation for Basic Research (project code 98-01-00260).

We denote by $C^m_\beta(\overline{\Omega})$ a weight space of $m$ times differentiable functions on the domain $\Omega$, $\|u\|_{C^m_\beta(\overline{\Omega})} = \max\limits_{|i|\le m}\max\limits_{x\in\overline{\Omega}} |\rho(x)^\beta D^i u(x)|$.

Let us cite the embedding theorems for the weight Sobolev spaces, which will be of use in what follows (see [19]).

**Theorem 1.** *Let* $1 < p < +\infty$, $\gamma = m - k - n/p > 0$, $k < m$, $\alpha, \beta \ge 0$. *If* $\beta < \alpha + \gamma$, *then* $W^m_{p,\beta}(\Omega)$ *is compactly embedded into* $C^k_\alpha(\overline{\Omega})$ *and the estimate holds*

$$\|u\|_{C^k_\alpha(\Omega)} \le c\rho^\tau_\Omega \|u\|_{W^m_{p,\beta}(\Omega)},$$

*where* $\tau = \max(0, \alpha - \beta)$.

**Theorem 2.** *Let* $1 < p \le q < +\infty$, $k < m$, $\gamma = m - k - n/p + n/q > 0$, $\alpha, \beta \ge 0$. *If* $\beta \le \alpha + \gamma$, *then* $W^m_{p,\beta}(\Omega)$ *is continuously embedded into* $W^k_{q,\alpha}(\Omega)$, *i. e.,*

$$\|u\|_{W^k_{q,\alpha}(\Omega)} \le c\|u\|_{W^m_{p,\beta}(\Omega)}.$$

*If* $\beta < \alpha + \gamma$, *then the embedding cited above is compact.*

**Theorem 3.** *Let* $\Gamma$ *be an* $n-1$*-dimensional smooth manifold. If* $\alpha < m - k - 1/p$, *then* $W^m_{p,\alpha}(\Omega)$ *is compactly embedded into* $W^k_p(\Gamma)$.

**2.** *Elliptic equation which degenerates on a part of boundary.* Consider the equation

$$Au = \sum_{k=0}^{m} \sum_{|i|,|j|=k} (-1)^k D^i(a_{i,j}(x) D^j u(x)) = f(x), \quad x \in \Omega. \tag{1}$$

We shall suppose that 1) $a_{ij} = a_{ji}$ and 2) the following conditions are fulfilled:

$$\overline{c}_m \rho^{2\alpha}(x) \sum_{|i|=m} |\xi_i|^2 \le \sum_{|i|,|j|=m} a_{i,j}(x)\xi_i\xi_j \le c_m \rho^{2\alpha}(x) \sum_{|i|=m} |\xi_i|^2, \tag{2}$$

$\overline{c}_m > 0$, $\alpha \ge 0$,

$$0 \le \sum_{|i|,|j|=k} a_{i,j}(x)\xi_i\xi_j \le c_k \rho^{2\alpha_k}(x) \sum_{|i|=k} |\xi_i|^2 \quad (x \in \Omega, \ \ \xi \in R^n), \tag{3}$$

$\alpha_k \ge 0$, $\alpha_k \ge \alpha - m + k$, $c_k \ge 0$, $k = \overline{0, m-1}$;

$$\int_\Omega |\rho^{-\alpha_0} f(x)|^2 dx. \tag{4}$$

With $\alpha > 0$ equation (1) degenerates on $S$.

We denote by $V$ the closure in the norm $W^m_{2,\alpha}(\Omega)$ of functions finite in $\Omega$. Everywhere in what follows we denote by $\|\cdot\|$ the norm of this space. The solution of equation (1) will be sought in the space $V$. We introduce on $V$ a bilinear form $a(u, v)$ and a linear functional $F(v)$, which are defined by the coefficients and the right-hand side of the equation

$$a(u, v) = \int_\Omega \sum_{|i|,|j|\le m} a_{i,j}(x) D^i u(x) D^j v(x) dx,$$

$$F(v) = \int_\Omega f(x) v(x) dx.$$

By a generalized solution of equation (1) in the space $V$ we shall call a solution $u \in V$ of the variational equation

$$a(u, v) = F(v) \quad \forall v \in V. \tag{5}$$

**3.** *Boundary conditions and resolvability.* Let us clarify how does the degeneration degree of equation $\alpha$ affect the boundary values of the solution and its derivatives and discuss in this connection the statement of the boundary conditions. First of all let us note that by virtue of the definition of the space $V$ the function $u \in V$ vanishes on $\partial\Omega \setminus S$ together with its derivatives up to $m-1$-st order

$$D^i u(x) = 0, \quad x \in \partial\Omega \setminus S, \quad |i| \le m-1.$$

Let $k$ stand for the most integer satisfying the inequality $k < m - 1/2 - \alpha$. Then (see Theorem 3) the function $u(x)$ and its derivatives up to the order $k$ inclusively (if $k \ge 0$) have the trace on $S$, which vanishes

$$D^i u(x) = 0, \quad x \in S, \quad |i| \le k.$$

For the derivatives $D^i u(x)$ of higher order $|i| > k$ the trace on $S$, generally speaking, is not defined, and in this case (for a smooth solution) on $S$ intrinsic boundary conditions arise. In particular, with the strong degeneration (i. e., for $\alpha \ge m - 1/2$), the trace operator on $S$ of the functions of the space $V$ is indefinite, and on $S$ only intrinsic boundary conditions arise.

Thus, if $m - k - 3/2 \le \alpha < m - k - 1/2$, then

$$V = \{v \in W^m_{2,\alpha}(\Omega) : D^i v(x) = 0,\ x \in S,\ |i| \le k; D^i v(x) = 0,\ x \in \partial\Omega \setminus S,\ |i| \le m-1\}.$$

In particular, if $\alpha \ge m - 1/2$, then

$$V = \{v \in W^m_{2,\alpha}(\Omega) : D^i v(x) = 0,\ x \in \partial\Omega \setminus S,\ |i| \le m-1\}.$$

If, in addition, $S = \partial\Omega$, i. e., $S$ equals the whole boundary of the surface of degeneration of equation's coefficients, then $V = W^m_{2,\alpha}(\Omega)$.

Now let us discuss the resolvability of problem (5). First we note that from the inequalities $\alpha_r \ge \alpha - m + r$ by virtue of Theorem 2 one has

$$\|u\|_{W^r_{2,\alpha_r}(\Omega)} \le c\|u\|_{W^m_{2,\alpha}(\Omega)}, \quad r = \overline{0, m-1}.$$

Using the conditions on the coefficients and the right-hand side, one can easily establish the continuity of both the form $a(u, v)$ and linear functional $F(v)$ on the space $V$ (below we write $\alpha_m = \alpha$)

$$a(u,v) \le \sum_{r=0}^{m} c_r |\rho^{\alpha_r}\nabla^r u|_{2,\Omega}\, |\rho^{\alpha_r}\nabla^r v|_{2,\Omega} \le c\|u\|\,\|v\|,$$

$$|F(v)| = \left| \int_\Omega f(x)v(x)dx \right| \le |\rho^{-\alpha_0} f|_{2,\Omega}\, |\rho^{\alpha_0} v|_{2,\Omega} \le c\|v\|.$$

By supposing that at least one of the conditions

$$\alpha < m - 1/2, \quad S \ne \partial\Omega, \quad a_{00} \not\equiv 0, \tag{6}$$

is fulfilled, we shall have the $V$-ellipticity of the form $a(u, v)$

$$a(u,u) \ge \overline{c}_m |\rho^\alpha \nabla^m u|^2_{2,\Omega} + \int_\Omega a_{00}(x)u^2(x)dx \ge c\|u\|^2.$$

Thus, by virtue of Lax–Milgram theorem, the following theorem takes place.

**Theorem 4.** *Problem* (5) *is uniquely resolvable in the space* $V$.

**4.** *Elliptic equation degenerating inside the domain.* Let a certain $n-1$-dimensional, sufficiently smooth surface $S \subset \overline{\Omega}$ divide $\Omega$ into the two domains $\Omega_1$ and $\Omega_2$. As above, $\rho(x)$ stands for the distance between $x$ and $S$. Consider equation (1) with input data satisfying conditions (2)–(4). Thus, coefficients of the equation degenerate on the surface $S$, i. e., inside the domain $\Omega$. In order to investigate such an equation and analyze its approximations by finite elements, we define on $\Omega$ a space of functions $W^m_{2,\alpha}(\Omega)$, which is a Sobolev space with the weight $\rho^\alpha(x)$, annihilating on $S$.

Let $Q_l u = u|_{\Omega_l}$ be the operators of restriction to $\Omega_l$ $(l = 1, 2)$, $k$ be the most integer satisfying the inequality $k < m - \alpha - 1/2$, $\mathbf{n}$ be the normal exterior to $\Omega_1$. Then

$$W^m_{2,\alpha}(\Omega) = \Big\{v \in L_{2,\alpha}(\Omega) : Q_l v \in W^m_{2,\alpha}(\Omega_l), \quad l = 1, 2; \quad \frac{\partial^r}{\partial \mathbf{n}^r} Q_1 v\Big|_S = \frac{\partial^r}{\partial \mathbf{n}^r} Q_2 v\Big|_S, \quad r = \overline{0, k}\Big\}$$

and

$$\|v\|_{W^m_{2,\alpha}(\Omega)} = \|Q_1 v\|_{W^m_{p,\alpha}(\Omega_1)} + \|Q_2 v\|_{W^m_{2,\alpha}(\Omega_2)}.$$

Note that in case $\alpha = 0$ the space defined above coincides with the classical (not weight) Sobolev space. The space $W^m_{2,\alpha}(\Omega)$ may be identified with a subspace of the Cartesian product $W^m_{2,\alpha}(\Omega_1) \times W^m_{2,\alpha}(\Omega_2)$, consisting of pairs of functions $(v_1, v_2) \in W^m_{2,\alpha}(\Omega_1) \times W^m_{2,\alpha}(\Omega_2)$, which satisfy the conjugacy conditions on $S$,

$$\frac{\partial^r}{\partial \mathbf{n}^r} v_1\Big|_S = \frac{\partial^r}{\partial \mathbf{n}^r} v_2\Big|_S, \quad r = \overline{0, k}.$$

Further, as in case of degeneration of a part of domain's boundary, we define a space $V$ as the closure in the norm of the space $W^m_{2,\alpha}(\Omega)$ of the set of functions which are finite in $\Omega$. Since by the condition the intersection of $S$ and $\partial\Omega$ has zero $n-1$-dimensional surface measure, we have

$$V = \Big\{v \in W^m_{2,\alpha}(\Omega) : \frac{\partial^r v}{\partial \mathbf{n}^r}\Big|_{\partial\Omega} = 0, \quad r = \overline{0, m-1}\Big\}$$

($\mathbf{n}$ is the exterior normal to $\partial\Omega$). By a generalized solution of equation (1) in the space $V$, as in Section 2, we shall mean a solution of the variational problem (5). Since the weight function $\rho(x)$ degenerates on the common part of the boundaries $\Omega_1$ and $\Omega_2$, for each of these subdomains the embedding Theorems 1–3 are valid. Consequently, arguing further as in Section 2, we establish the resolvability of the problem.

**Theorem 5.** *Problem* (5) *in the case of degenerating coefficients under consideration has the unique solution.*

Further, let us clarify what are conditions, to which the solution of problem (5) satisfies at the points of degeneration. By integrating by parts in each separate domain $\Omega_l$, combining results, and denoting by $[v]_S$ the jump of $v$ in transition over $S$, after some transformations we get

$$a(u, v) = \int_\Omega \Big(\sum_{k=0}^{m} \sum_{|i|,|j|=k} (-1)^k D^i(a_{i,j} D^j u)\Big) v\, dx +$$
$$+ \int_S \sum_{r=0}^{k} [q_{2m-1-r}(u)]_S \frac{\partial^r v}{\partial n^r} dx + \int_S \sum_{r=k+1}^{m-1} q_{2m-1-r}(u) \frac{\partial^r v}{\partial n^r} dx,$$

where $q_r(u)$, $r = \overline{m, 2m-1}$, are differential expressions which arise on $S$ within the integration by parts (for $m = 2$, below we give their explicit form; in the general case, we omit these expressions since they are cumbersome). Thus, we get the following conditions on the surface of degeneration $S$:

$$\Big[\frac{\partial^r u}{\partial n^r}\Big]_S = 0, \quad [q_{2m-1-r}(u)]_S = 0, \quad r = 0, 1, \dots, k, \tag{7}$$

$$q_r(u)|_S = 0, \quad r = m, m+1, \dots, 2m-k-2. \tag{8}$$

Let us mark two extremal cases.

1. $\alpha < 1/2$. In this case, we have $k = m - 1$ and conditions (7) are fulfilled; conditions (8) are absent.
2. $\alpha \geq m - 1/2$. In this case, we have $k < 0$ and intrinsic conditions (8) are fulfilled; conditions (7) are absent. Therefore problem (5) in fact breaks up into two independent problems in the subdomains $\Omega_1$ and $\Omega_2$ with intrinsic boundary conditions on $S$ which is the common part of these boundaries (see the definition of the spaces $W^m_{2,\alpha}(\Omega)$ and $V$ given above).

We give the explicit form of the differential expressions $q_r(u)$, where $r = 2, 3$, for equations of fourth order ($m = 2$) in a two-dimensional ($n = 2$) domain. Write

$$A_j = \sum_{|i|\le 2} a_{ij} D^i u, \quad |j| \le 2.$$

Then

$$q_2(u) = A_{20}\cos^2(\mathbf{n}, x_1) + A_{11}\cos(\mathbf{n}, x_1)\cos(\mathbf{n}, x_2) + A_{02}\cos^2(\mathbf{n}, x_2),$$

$$q_3(u) = \Big(A_{10} - \frac{\partial}{\partial x_1}A_{20} - \frac{\partial}{\partial x_2}A_{11}\Big)\cos(\mathbf{n}, x_1) + \Big(A_{01} - \frac{\partial}{\partial x_2}A_{02} - \frac{\partial}{\partial x_1}A_{11}\Big)\cos(\mathbf{n}, x_2) - \\ - \frac{\partial}{\partial \boldsymbol{\tau}}((A_{20} - A_{02})\cos(\mathbf{n}, x_1)\cos(\mathbf{n}, x_2) + A_{11}(\cos^2(\mathbf{n}, x_1) - \cos^2(\mathbf{n}, x_2)))$$

($\boldsymbol{\tau}$ is the tangent vector to $S$).

**5.** *Finite element method.* In what follows in description of FEM we use terminology and notation adopted in [21]. Here we restrict ourselves to consideration of degeneracy inside of a two-dimensional domain for equation of fourth order. In what follows for the sake of simplicity we shall assume that the domain $\Omega$ is a polygon, and the degeneracy line $S$ is piecewise linear. Let $\Omega$ be correctly regularly triangulated (divided into triangular or rectangular finite elements). Assume that its triangulation $T_h$ is such that $S$ does not intersect the interiority of any finite element participating in the triangulation. Suppose that a finite-dimensional space of approximate solutions $V_h$ satisfies the condition

$$V_h \subset C^1(\Omega) \bigcap \overset{o}{W}{}_2^2(\Omega)$$

(examples of such finite-dimensional spaces can be found in [21], pp. 325–347). Under fulfillment of these conditions the conformity of the method is ensured, i. e., the inclusion $V_h \subset V$.

By an approximate solution of problem (5), as usual, we shall understand a function $u_h \in V_h$ which satisfies the integral identity

$$a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h. \tag{9}$$

Conditions (2), (3) ensure unique resolvability of problem (9) for any $f \in (W^2_{2,\alpha}(\Omega))^*$ (in particular, for the right side $f(x)$, satisfying condition (4)) and the validity of the estimate for the difference between the exact and approximate solutions

$$\|u - u_h\| \le c \inf_{v_h \in V_h} \|u - v_h\| \tag{10}$$

(see, e. g., [21], p. 315).

Now we turn to estimation of $\inf\limits_{v_h \in V_h} \|u - v_h\|$ via $\|u - \Pi_h u\|$, where $\Pi_h$ is the operator of $V_h$-interpolation, for some special forms of finite elements. In its turn, the estimate of the quantity (global estimate of interpolation error) $\|u - \Pi_h u\|$ is a result of combining of errors of finite elements' (which participate in the interpolation process) interpolation. Thus, the analysis of the FEM accuracy is reduced to the estimation of interpolation errors at weight norms on a typical finite element.

We introduce notation

$$\rho_K = \sup\{\rho(x) : x \in K\}, \quad h_K = \operatorname{diam} K, \\ h = \max\{h_K : K \in T_h\}, \quad d_k = \sup\{\operatorname{diam} S : S \subset K, \quad S \text{ is a circle}\}.$$

Obviously, $d_K \le h_K$. The following regularity condition is assumed to be fulfilled $h_K \le \sigma d_K$. Let $(K, P_K, \Sigma_K)$ be a typical element of the triangulation $T_h$, $l$ be the maximal order of derivatives participating in the determination of the set of freedom degrees $\Sigma_K$ of the element $K$, $\Pi_K : C^l(K) \to P_K$ being the operator of $P_K$-interpolation. In the formulation of estimates we shall use notation

$$W^{m+1}_{p,\beta} \overset{c}{\subset} W^r_{q,\alpha} \bigcap C^l, \tag{11}$$

which ensures the fulfillment of the following inequalities:

$$p \le q, \quad \alpha, \beta \ge 0, \quad r < m+1, \quad \gamma \equiv m+1-2/p-r+2/q > 0,$$
$$\beta < \alpha + \gamma, \quad \beta < m+1-2/p,$$

which ensure the compactness of the embedding (see Theorems 1, 2)

$$W^{m+1}_{p,\beta}(K) \overset{c}{\subset} W^r_{q,\alpha}(K) \bigcap C^l(K)$$

for arbitrary $K \in T_h$. By using the technique of [20], for affine families of finite elements (either Lagrangian or Hermitian, see [21], p. 88) we establish the following basic estimate.

**Theorem 6.** *If $T_h$ is affine and inclusion* (11) *takes place, then a constant $c$ exists, independent of $S$, such that*

$$\|u - \Pi_K u\|_{W^r_{q,\alpha}(K)} \le c\rho_K^{\alpha-\beta} h_K^{\gamma} |\rho^\beta \nabla^{m+1} u|_{p,K},$$

*where $\gamma = m+1-2/p-r+2/q$.*

Let us note that these estimates generalize to weight norms well-known (and turned classical) estimates of finite element interpolation in the non-weight Sobolev spaces (see, e. g., [21]).

Using these general results, we get estimates of interpolation error on some concrete finite elements.

1) Hermitian rectangular Bogner–Fox–Schmidt element (see [21], p. 82). In this case, $m = 3$, $\gamma = 4 - 2/p - r + 2/q$ $(r < 4)$, and with $\beta < \gamma + \alpha$, $\beta < 2 - 2/p$ we have $W^4_{p,\beta} \subset C^2$. The following estimate takes place

$$\|u - \Pi_K u\|_{W^r_{q,\alpha}(K)} \le c h_K^{4-r-2/p+2/q} \rho_K^{\alpha-\beta} |\rho^\beta \nabla^4 u|_{p,K}. \tag{12}$$

2) Hermitian triangle of the type (5) (see [21], p. 329). In this case, $P_K = P_5(K)$, $\dim P_K = 21$, $m = 5$, $\gamma = 6 - 2/p - r + 2/q$ $(r < 6)$, $\beta < 4 - 2/p$, and with $\beta < \gamma + \alpha$ we have $W^6_{p,\beta} \subset C^2$. The following estimate takes place

$$\|u - \Pi_K u\|_{W^r_{q,\alpha}(K)} \le c h_K^{6-r-2/p+2/q} \rho_K^{\alpha-\beta} |\rho^\beta \nabla^6 u|_{p,K}. \tag{13}$$

Similar estimates can be found also for so-called almost affine families of finite elements, which are frequently used in solving fourth order equations.

Consider, for example, the Argyros triangle (see [21], p. 328).

**Theorem 7.** *Let $r < 6$, $1 < p \le q \le +\infty$, $\gamma = 6 - 2/p - r + 2/q$, $\beta < \gamma + \alpha$, $\beta < 4 - 2/p$. Then*

$$\|u - \Pi_K u\|_{W^r_{q,\alpha}(K)} \le c h_K^{\gamma} \rho_K^{\alpha-\beta} |\rho^\beta \nabla^6 u|_{p,K}. \tag{14}$$

**Proof** (cf. [21], p. 330). Let $\Lambda : C^2(K) \to P_5(K)$ be interpolation operator on Hermitian triangle of the type (5). By virtue of (13),

$$\|u - \Lambda u\|_{W^r_{q,\alpha}(K)} \le c h_K^{\gamma} \rho_K^{\alpha-\beta} |\rho^\beta \nabla^6 u|_{p,K}.$$

Let us estimate the function $\eta = \Pi_K u - \Lambda u$ in the norm $W^r_{q,\alpha}(K)$. Obviously, $\eta(x) = 0$ for $x \in \partial K$, because $D^i \eta(a_s) = 0$ for $|i| \le 2$. In addition,

$$\eta'(b_s)(a_s - b_s) \equiv \nabla \eta(b_s)(a_s - b_s) = \frac{\partial}{\partial n}(u - \Lambda u)(b_s)(a_s - b_s) n_s$$

for $s = 1, 2, 3$ (see [21], p. 330).

Let $\eta_s$ be a basis function corresponding to the freedom degree $v \to v'(b_s)(a_s - b_s)$. Then

$$\eta = \Pi_K u - \Lambda u = \sum_{s=1}^{3} \eta'(b_s)(a_s - b_s)\eta_s = \sum_{s=1}^{3} \frac{\partial}{\partial n}(u - \Lambda u)(b_s)[(a_s - b_s)n_s]\eta_s.$$

Since $\left|\frac{\partial}{\partial n}(u - \Lambda u)(b_s)\right| \le \sqrt{2}\|u - \Lambda u\|_{W^1_\infty(K)}$, by Theorem 6 with $r = 1$, $q = \infty$, $m = 5$, $\gamma = 5 - 2/p$, and $\alpha = 0$ we get $\|u - \Lambda u\|_{W^1_\infty(K)} \le ch_K^{5-2/p}\rho_K^{-\beta}|\rho^\beta\nabla^6 u|_{p,K}$. Further, obviously, for almost all $x$ in $K$ and for $r = 0, 1, \ldots, r$ we have $|\rho^\alpha(x)\nabla^r\eta_s(x)| \le \rho_K^\alpha|\nabla^r\eta_s(x)|$, whence $|\rho^\alpha\nabla^r\eta_s(x)|_{q,K} \le \rho_K^\alpha|\nabla^r\eta_s(x)|_{q,K}$. Since (see [21], p. 331) $|\nabla^r\eta_s(x)|_{q,K} \le h_K^{2-2/q-r}|\nabla^r\widehat{\eta}_s|_{q,\widehat{K}}$, with regard for the estimate $|(a_s - b_s)\cdot n_s| \le h_K$ we shall finally have

$$\|\Pi_K u - \Lambda u\|_{W^r_{q,\alpha}(K)} \le ch_K^{5-2/p}\rho_K^{-\beta}h_K h_K^{2/q-r}\rho_K^\alpha|\rho^\beta\nabla^6 u|_{p,K} = ch_K^{6-2/p+2/q-r}\rho_K^{\alpha-\beta}|\rho^\beta\nabla^6 u|_{p,K}. \quad \square$$

Estimates (12)–(14) show that the interpolation error on an element depends on both the element's diameter and the distance between the element and the degeneration set $S$. Therefore, in estimation of errors in the whole domain it seems natural to make the quantities $\rho_K$ and $h_K$ consistent. Namely, in following [20], [15]–[17], we shall say that a family of triangulations concentrates near the set $S$ with the concentration degree $\mu \ge 1$ if positive constants $\sigma_1$, $\sigma_2$ exist, independent of $h$, such that for all finite elements of the triangulation the inequality holds:

$$\sigma_1 h^\mu \le h_K \le \sigma_2 h\rho_K^{1-1/\mu}. \tag{15}$$

Condition (15) means that a finite element, situated on a distance $O(1)$ from $S$, has the diameter $O(h)$, and for elements which lie near $S$, we have $h_K = O(h^\mu)$. In case $\mu = 1$ condition (15) means quasi-uniformity of the triangulation.

Now, by assuming that the solution of problem (5) satisfies the corresponding conditions of smoothness and using estimates (10), (12)–(14), one can get (in just the same way as in [15]–[17]) estimates of FEM accuracy in using a concrete finite element. Let us formulate, for example, the corresponding result for the Argyros triangle.

**Theorem 8.** *Let $u$ be a solution of problem* (5), *such that its restriction to each of the subdomains $\Omega_l$, $l = 1, 2$, belongs to the class $W^6_{2,\beta}(\Omega_l)$. Assume that the conditions of Theorem* 7 *are fulfilled for $r = 2$, $p = q = 2$, as well as conditions* (2), (3) *and the inequality* (15). *Then*

$$\|u - u_h\|_{W^2_{2,\alpha}(\Omega)} \le ch^\vartheta|\rho^\beta\nabla^6 u|_{2,\Omega},$$

*where $\vartheta = \min(\gamma, \mu(\gamma + \alpha - \beta))$.*

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27 January 1999 Kazan State University