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STABILITY OF THE EQUATIONS WITH DELAY. II

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In the previous articles (see [1], [2]), basing of the ideas of the theory of an "abstract functional differential equation" (see [3]–[5]), we suggested a new concept of the stability for linear ordinary differential equations and their generalizations. The concept mentioned below and the results in [6]–[9] are developed for quasilinear equations.

Denote by $\mathbf{C}$ the Banach space of continuous and bounded functions $x : [0,\infty) \to \mathbf{R}^n$ with the norm $\|x\|_{\mathbf{C}} \stackrel{\text{def}}{=} \sup_{t\geq 0} |x(t)|$, where $|\cdot|$ stands for the norm in the space $\mathbf{R}^n$. Further, let the space $\mathbf{C}_0$ and the *weight* space $\mathbf{C}_\gamma$ for $\gamma > 0$ be defined via the equalities

$$\mathbf{C}_0 \stackrel{\text{def}}{=} \{x \in \mathbf{C} : \lim_{t\to\infty} |x(t)| = 0,\ \|x\|_{\mathbf{C}_0} \stackrel{\text{def}}{=} \|x\|_{\mathbf{C}}\}, \quad \mathbf{C}_\gamma \stackrel{\text{def}}{=} \{x \in \mathbf{C} : \|x\|_{\mathbf{C}_\gamma} \stackrel{\text{def}}{=} \sup_{t\geq 0} e^{\gamma t}|x(t)| < \infty\}.$$

The spaces $\mathbf{C}_0$ and $\mathbf{C}_\gamma$ are Banach.

The Cauchy problem

$$\dot{x}(t) = f(t, x(t)),\ x(0) = \alpha, \quad t \geq 0, \tag{1}$$

is equivalent to the equation

$$x(t) = \int_0^t f(s, x(s))\, ds + \alpha. \tag{2}$$

Therefore the definitions of the stability of the equation $\dot{x}(t) = f(t, x(t))$ given in [10] (p. 9) in the case where $f(t,0) \equiv 0$, can be reformulated as follows: The trivial solution of problem (1) is stable by Lyapunov (asymptotically or exponentially) if equation (2) has in the space $\mathbf{C}$ (in the spaces $\mathbf{C}_0$ or $\mathbf{C}_\gamma$, respectively) a unique solution for each $\alpha \in \mathbf{R}^n$ and this solution depends continuously on α in the metric of this space. In other words, this type of stability is the correct resolvability of equation (2) in a given space. In [10], Krasovskiĭ considered also a "stability with respect to constantly acting perturbations η", which is the continuous dependence of a solution of the problem

$$\dot{x}(t) = f(t, x(t)) + \eta(t),\ x(0) = \alpha, \quad t \geq 0,$$

on the perturbations η.

Now let us consider a general functional differential equation $\mathcal{L}x = \mathcal{F}x$ under assumption that $\mathcal{L} : \mathbf{D} \to \mathbf{L}$ is linear and $\mathcal{F} : \mathbf{D} \to \mathbf{L}$ is nonlinear Volterra operators (see [3], [4]). Here $\mathbf{L}$ is a linear space of classes of equivalent locally summable functions $z : [0,\infty) \to \mathbf{R}^n$, $\mathbf{D}$ is a linear space of all functions absolutely continuous on each finite segment $x : [0,\infty) \to \mathbf{R}^n$. Further, let a Banach space $\mathbf{B}$ with the norm $\|\cdot\|_{\mathbf{B}}$ be given, whose elements belong to the space $\mathbf{L}$, and a solution $x \in \mathbf{D}$ of the Cauchy problem $\mathcal{L}_0 x = z$, $x(0) = \alpha$ for the "model" equation $\mathcal{L}_0 x = z$ with a linear Volterra

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