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TOTAL AND CO-TOTAL DEGREES OF ENUMERABILITY

B.Ya. Solon

Let ω be the set of natural numbers, $A, B, C, \dots, X, Y, Z \subseteq \omega$ (with or without indices), $\overline{A} = \omega \setminus A$; let $c_A(x)$ be the characteristic function of the set A: $c_A(x) = 1$ if $x \in A$, and $c_A(x) = 0$ if $x \in \overline{A}$. Let $\langle x, y\rangle$ stand for the Cantor index of the ordered pair (x, y), if $z = \langle x, y\rangle$ then $\langle z\rangle_1 = x$ and $\langle z\rangle_2 = y$; in addition, $\langle A\rangle_1 = \{x : \exists y[\langle x, y\rangle \in A]\}$ and $\langle A\rangle_2 = \{y : \exists x[\langle x, y\rangle \in A]\}$. As usual, D_u stands for a finite set with the canonical index u, we use the symbol D as a variable for finite sets. For a finite set A the symbol $|A|$ denotes the number of elements of A; $|A| < \infty$ if A is a finite set. Denote the total functions acting from ω to ω by f, g, h (with or without indices). If α is a partial function then $\delta\alpha$ stands for its domain and $\rho\alpha$ does for its range; $\tau\alpha = \{\langle x, y\rangle : x \in \delta\alpha \,\&\, y = \alpha(x)\}$ is the graph of the function α. We write $\alpha \subseteq \beta$ if $\tau\alpha \subseteq \tau\beta$. A set A is called *univalent* if $A = \tau\alpha$ for a certain function α. Using the notation of [1] (p. 39), denote by W_t the t-th computably enumerable (c. e.) set, $K = \{t : t \in W_t\}$ and $K_0 = \{\langle x, t\rangle : x \in W_t\}$.

A set A is *enumeration reducible* (*e-reducible*) to a set B (symbolically, $A \leq_e B$) if one can obtain an enumeration of A using any enumeration of B by a certain uniform algorithm. Formally, (see [2])

$$A \leq_e B \iff \exists t\, \forall x[x \in A \iff \exists u[\langle x, u\rangle \in W_t \,\&\, D_u \subseteq B]].$$

Instead of $\tau\alpha \leq_e A$ we write $\alpha \leq_e A$. Denote by $\Phi_t : 2^\omega \to 2^\omega$ the *t-th e-operator* such that for any X,

$$\Phi_t(X) = \{x : \exists u[\langle x, u\rangle \in W_t \,\&\, D_u \subseteq X]\}.$$

Consequently, $A \leq_e B \iff \exists t[A = \Phi_t(B)]$. It is well known [2] that e-operators are monotonous and continuous, i. e., for any t and any X, Y,

$$X \subseteq Y \to \Phi_t(X) \subseteq \Phi_t(Y)$$

and

$$\forall x[x \in \Phi_t(X) \to (\exists D)[D \subseteq X \,\&\, x \in \Phi_t(D)]].$$

Let, as usual, $A \equiv_e B \iff A \leq_e B \,\&\, B \leq_e A$; let $d_e(A) = \{X : X \equiv_e A\}$ denote the *degree of enumerability* (*e-degree*) of the set A, and $d_e(A) \leq d_e(B) \iff A \leq_e B$. We also denote the e-degrees using the lower case bold Latin letters $\mathbf{a}, \mathbf{b}, \dots$ (with or without indices). If e-degrees $\mathbf{a}$ and $\mathbf{b}$ are incomparable with respect to $\leq$ then we write $\mathbf{a}|\mathbf{b}$, and we do $\mathbf{a} < \mathbf{b}$ if $\mathbf{a} \leq \mathbf{b} \,\&\, \mathbf{b} \not\leq \mathbf{a}$.

Let $\mathbf{D}_e$ stand for the set of e-degrees partially ordered by the relation $\leq$. It is well known (see, e. g., [3]) that $\mathbf{D}_e$ is the upper semilattice with the least element $\mathbf{0}_e = d_e(\emptyset) = d_e(K)$, it is not a lattice, where the least upper bound $\mathbf{a} \vee \mathbf{b}$ of any two e-degrees $\mathbf{a} = d_e(A)$ and $\mathbf{b} = d_e(B)$ equals $\mathbf{a} \vee \mathbf{b} = d_e(A \oplus B)$; $A \oplus B = \{2x : x \in A\} \cup \{2x + 1 : x \in B\}$. Let $\mathbf{D}_e(\leq \mathbf{a}) = \{\mathbf{x} : \mathbf{x} \leq \mathbf{a}\}$.

In [4], the e-jump $\mathbf{J}(A)$ of a set A is defined as follows:

$$\mathbf{J}(A) = \tau c_{A^e} \equiv A^e \oplus \overline{A^e},$$

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