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Commutator Inequalities Associated with Polar Decompositions of τ-Measurable Operators

D. Dauitbek*, N. E. Tokmagambetov, and K. S. Tulenov*****

al-Farabi Kazakh National University, pr. al-Farabi 71, Almaty, 050040 Republic of Kazakhstan

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Abstract—We prove commutator inequalities associated with polar decompositions of τ-measurable operators.

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1. INTRODUCTION

Let $L_0(\mathcal{M})$ be a space of τ-measurable operators. Then there exist a partially measurable operator $U \in \mathcal{M}$ and a positive semidefinite τ-measurable operator $|A|$ for any operator $A \in L_0(\mathcal{M})$. These operators meet the relation

$$A = U|A|. \tag{1}$$

Relation (1) is a so-called polar decomposition of the operator A. The semi-positive definite part of this decomposition, i.e., the modulus of the operator $|A|$, is unique and $|A| = (A^*A)^{1/2}$. Spectral theory tells us that the operator A is normal (i.e., $A^*A = AA^*$) if and only if $U|A| = |A|U$. Or, equivalently the conjugate commutator $A^*A - AA^*$ vanishes if and only if the commutator $U|A| - |A|U$ also vanishes [1]. We note the works [2–5] which contain new necessary and sufficient conditions on the projections commutation in the terms of operator inequalities on von Neumann algebras. The commutator inequalities relative to the polar decompositions for any unital norms and $(n \times n)$-complex matrices can be found in [6]. This work extends some results of [6] to the τ-measurable operators and $\|\cdot\|_{E(\mathcal{M})}$-norms. The principal result of the paper is Lemma 3.4.

Since the technique of paper [6] cannot be applied to extend the results of the paper to τ-measurable operators we need certain auxiliary notions given below.

2. PRELIMINARY NOTIONS

Let $\mathcal{M}$ be a finite von Neumann algebra in the Hilbert space $\mathcal{H}$ with the faithful normal semifinite trace τ, $\mathcal{M}^{\mathrm{pr}}$ be the projections lattice of the algebra $\mathcal{M}$, I be the unit of the algebra $\mathcal{M}$ and $P^{\perp} = I - P$ for $P \in \mathcal{M}^{\mathrm{pr}}$. Any closed densely defined in $\mathcal{H}$ operator T with the domain $D(T)$ is said to be affiliate with $\mathcal{M}$ ($T\eta\mathcal{M}$) if and only if $U^*TU = T$ for all unital U which belong to the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. The operator $T\eta\mathcal{M}$ is called τ-measurable if for any $\varepsilon > 0$ there exists a projection $P \in \mathcal{M}$ such that $\mathcal{PH} \subseteq D(T)$ and $\tau(P^{\perp}) < \varepsilon$. Let us denote the set of all such τ-measurable operators by $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is a $*$-algebra with strong sum and product which are in fact the closures of the algebraic sum and product.

*E-mail: dostilek.dautbek@gmail.com.
**E-mail: niyaz.tokmagambetov@gmail.com.
***E-mail: tulen.kz@mail.ru.