ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2011, Vol. 55, No. 8, pp. 19–27. © Allerton Press, Inc., 2011.
Original Russian Text © M.Yu. Kokurin, 2011, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2011, No. 8, pp. 23–33.

# An Exact Penalty Method for Monotone Variational Inequalities and Order Optimal Algorithms for Finding Saddle Points

**M. Yu. Kokurin**[1*]

[1] *Mari State University, pl. Lenina 1, Yoshkar-Ola, 424000 Russia*

Received May 7, 2010

**Abstract**—We consider variational inequalities in a Banach space. We propose an exact penalty method which enables one to remove functional constraints. The obtained result is used for constructing optimal (in the sense of complexity) iterative schemes for finding saddle points under functional constraints.

**DOI:** 10.3103/S1066369X11080044

Keywords and phrases: *variational inequality, monotone operator, Banach space, exact penalty, saddle point, complexity.*

**1.** Consider a variational inequality (VI) in the form

$$x \in Q : \langle F(x),\, x - z\rangle \le 0 \ \ \forall z \in Q. \tag{1}$$

Here $Q$ is a convex closed set in a real reflexive Banach space $X$, while $F : D_F \subset X \to 2^{X^*}$ is a monotone point-set mapping to the conjugate space $X^*$. We assume that $Q \subset \operatorname{int} D_F$. By definition, the monotonicity of the mapping $F$ means that

$$\langle f(x_1) - f(x_2),\, x_1 - x_2\rangle \ge 0 \ \ \forall\, x_1, x_2 \in D_F;\ \ f(x_1) \in F(x_1),\ \ f(x_2) \in F(x_2). \tag{2}$$

Hereinafter for a functional $f \in X^*$ and an element $x \in X$ the symbol $\langle f, x\rangle$ stands for the value of $f$ on $x$. If $X$ is a Hilbert space, then $X^*$ is canonically identical to $X$, and the symbol $\langle f, x\rangle$ stands for the scalar product of elements $f, x \in X$. In addition, we assume that $F$ is the maximal monotone mapping, i.e., no other monotone mapping exists whose graph contains that of $F$ as a proper part. Very important examples of such mappings are single-valued hemicontinuous monotone operators ([1], P. 64) defined on the whole space $X$, as well as subdifferential mappings generated by proper convex and convex-concave functionals ([2], pp. 149, 153).

We understand a solution to VI (1) as a point $x_* \in Q$ such that some element $f(x_*) \in F(x_*)$ satisfies the correlation

$$\langle f(x_*),\, x_* - z\rangle \le 0 \ \ \forall z \in Q. \tag{3}$$

In what follows we assume that the set $Q$ takes the form

$$Q = Q_0 \cap Q_1,\ \ Q_1 = \bigl\{x \in X : \varphi_j(x) \le 0,\ j \in J = \{1, 2, \dots, m\}\bigr\}, \tag{4}$$

where $Q_0$ is a convex closed set in $X$, while $\varphi_j$, $j \in J$, are convex lower semicontinuous functionals. We assume that

$$Q_0 \subset \operatorname{int} D_F \bigcap \bigcap_{j\in J} \operatorname{int} \operatorname{dom} \varphi_j.$$

Thus, functionals $\varphi_j$, $j \in J$, are locally Lipschitz and subdifferentiable on $Q_0$ ([2], P. 113; [3], pp. 197–198). Many problems of mathematical physics, optimization theory, and operations research [1–6] are reducible to the form (1). Among them let us mention the convex programming problem

$$\min\{\varphi_0(x) : x \in Q\} \tag{5}$$

*E-mail: kokurin@marsu.ru.