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The Linear Conjugation Problem for Bi-Analytic Functions

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Abstract—We consider a classical problem on linear conjugation problem for bi-analytic functions on smooth contour. We obtain explicit formula of a solution to a problem and describe necessary and sufficient conditions of its solvability.

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Let an oriented smooth contour Γ consisting of simple curves $\Gamma_1, \dots, \Gamma_m$ be defined on the complex plane. Then its complement $D = \mathbb{C} \setminus \Gamma$ consists of several domains $D_0, D_1, \dots, D_m$ such that the domain D_0 is infinite and contains a neighborhood of the point at infinity ∞, and all other domains are finite. We consider in these domains a bi-analytic function ϕ, i.e., a function $\phi \in C^2(D)$ satisfying the equation

$$\frac{\partial^2 \phi}{\partial \overline{z}^2} = 0.$$

It is well-known [1, 2], that it allows representation in terms of two analytic functions ϕ_0, ϕ_1 by the Goursat formula

$$\phi(z) = \phi_0(z) + \overline{z}\phi_1(z), \quad z \in D, \tag{1}$$

where $\phi_1 = \partial\phi/\partial\overline{z}$.

Let bi-analytic in D function ϕ together with its partial derivative $\phi_1 = \partial\phi/\partial\overline{z}$ be continuous in closed domains $\overline{D}_j$, i.e., it has one-side boundary values $\phi^\pm$ and $\phi_1^\pm$ on Γ. Then there is determined a problem on linear conjugation

$$\phi^+ - G_0\phi^- = f_0, \quad \left(\frac{\partial\phi}{\partial\overline{z}}\right)^+ - G_1\left(\frac{\partial\phi}{\partial\overline{z}}\right)^- = f_1, \tag{2}$$

where coefficients G_k and right-hand parts f_k are given, and $G_k(t) \neq 0$ for any $t \in \Gamma$.

In what follows we assume that the functions G_k and f_k belong to the Hölder class $C^\mu(\Gamma)$, and solutions belong to the class

$$\phi, \phi_1 \in C^\mu(\overline{D}_j), \ 1 \le j \le m; \quad \phi, \phi_1 \in C^\mu(\overline{D}_0 \cap \{|z| \le R\}), \tag{3}$$

where $\phi_1 = \partial\phi/\partial\overline{z}$, and $R > 0$ is selected so that $\Gamma \subseteq \{|z| < R\}$. Additionally, for certain integer k the behavior of ϕ at infinity point satisfies the bound

$$|\phi(z)| + |z|\,|\phi_1(z)| \le C|z|^{k-1} \ \text{for} \ |z| \ge R. \tag{4}$$

The problems of similar type are studied by a number of authors (e.g., [3, 4]), as a rule, for functions bounded at the infinity point. The scheme of its solving is well-known [5]. It reduces to certain problems of analogous type for analytic functions by means of representation (1). But in general case of arbitrary

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