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On Complexity of Problem of Satisfiability for Systems of Countable-Valued Functional Equations

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Abstract—We consider the problem of satisfiability for systems of countable-valued functional equations, containing ternary discriminator function p. We prove that this problem is m-complete in the class Π_1 of Kleene–Mostowski hierarchy.

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Functional equations find a broad range of applications in virtually all areas of Mathematics. A distinctive feature of functional equations is the fact that functions and collections of functions are considered as the solutions of these equations, while the subject variables in the equations are under the universal quantifier and in essence only "outline" the main subject area. The expressive capabilities of the language of functional equations are much richer than those of the language of equations without functional variables. This, to a large extent, explains the fact that functional equations are so widespread.

In discrete mathematics, the systematic investigation of functional equations has begun relatively recently. Functional Boolean equations and functional equations of many-valued logic are studied in [1–10]. Works [4–9], in particular, give a comprehensive answer to the question of definability of the sets of functions by systems of functional equations over arbitrary sets of functional constants. Let us note that the problem of satisfiability for the systems of functional equations of many-valued logic can be solved by exhaustive search algorithms, but not by polynomial algorithms [10].

In the transition from functional equations of many-valued logic to functional equations of countable-valued logic, a qualitative leap occurs: the majority of the considered problems are far beyond algorithmic efficiency [11, 12]. Thus, it makes sense to consider the problem of satisfiability for the systems of functional equations of countable-valued logic only for those systems of equations that either contain no functional constants at all, or contain "very weak" functional constants. A ternary discriminator function p can be classified as such functional constant, i.e., a function well-known in the universal algebra, the theory of functions of many-valued logic, and the theory of recursive functions [13–15].

The choice of the function p is conditioned by the following arguments. First, the discriminator p is a homogeneous function [13], i.e., a maximally "symmetric" function (a functional with a maximal possible group of automorphisms). Hence, by means of p, one can neither "pick" a concrete natural number, nor define any of the known arithmetical functions. Second, the discriminator p allows to "discriminate" unequal natural numbers in the framework of systems of functional equations. This property allows to pass, by means of p, from the language of functional equations to a richer language of QFEC, that includes a complete system of logical connectives and quantifiers with respect to subject variables [11]. The QFEC language turns out to be the most convenient mean for construction of the formulas related to the considered problem of satisfiability. Hence, we use it in the proof of unsolvability of the problem of satisfiability for the systems of functional equations with a functional constant p.

As we mentioned above, the problem of satisfiability for systems of functional equations of many-valued logic is algorithmically solvable. The method by which the main result of [10] was obtained demonstrates that by the systems of functional equations one can model computations on sufficiently powerful computation facilities. In case of countable-valued logic and in the presence of functional

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