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# GEODESIC FLOW ON THE SPHERICAL TANGENT BUNDLE OF TWO-DIMENSIONAL MANIFOLD WITH THE SASAKIAN METRIC

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As is well-known, the Hamiltonian equations constitute one of the most important classes of the differential equations. In particular, these equations appear in the problem on finding geodesics in Riemannian manifolds. At the same time, the completely integrable Hamiltonian systems rarely occur (see [1]–[4]). In this article we consider the geodesic flow on a spherical tangent bundle of a two-dimensional manifold endowed with the Sasakian metric. We demonstrate that, in this case where the basic manifold is locally isometric to a surface of revolution, the corresponding Hamiltonian system is completely integrable in the Liouville sense (i. e., the two independent first integrals exist). Using this fact, we find the flow trajectories in the form of integrals.

This investigation arose from the author's study of variational problems for rotation functional of curves (see [5]–[9]). In this article, with the use of the results by P. Nagy (see [10]), we demonstrate that the basic flow trajectories (i. e., the projections of flow trajectories onto the base) are the (first order) isoperimetric rotation functionals. We find a generalized Clairaut integral for these extremals, which turns out to be the missed integral required for the complete integrability of the geodesic flow on the spherical tangent bundle endowed with the Sasakian metric (see [11], [12]) in case the base manifold is locally isometric to a surface of revolution. Note that the existence of the classic Clairaut integral implies the complete integrability of the geodesic flow on a surface of revolution (see [1], [2]). The author is greatly indebted to A.T. Fomenko, who conjectured that the generalized Clairaut integral plays such a role in the theory of geodesic flow on bundle; this conjecture was completely confirmed.

## 1. Isoperimetric rotation extremals on two-dimensional Riemannian manifolds

On the space of parameterized curves in a Riemannian manifold $(M^n, g)$ we consider the two functionals: the length functional $l$ and the functional of absolute rotation $\theta$. For a parameterized curve $\gamma$, we have $l[\gamma] = \int_{t_1}^{t_0} (g_{ij}\dot{x}^i\dot{x}^j)^{1/2} dt$, where $\dot{x}^i = \xi^i$ are coordinates of the tangent vector $\dot{\gamma}$, and $\theta[\gamma] = \int_{l_1}^{l_0} k_g dl$, where $l$ is the arc length of $\gamma$, while $k_g$ stands for the first Frenet curvature of $\gamma$ (if $n = 2$, $k_g$ is the absolute geodesic curvature).

For the rotation functional we consider the isoperimetric variational problem with fixed end-points:

$$\delta\theta = 0, \ \ \gamma(t_0) = p_0, \ \ \gamma(t_1) = p_1, \ \ l_0 = 0, \ \ l_1 = \widehat{l}, \ \ l[\gamma] = \widehat{l} = \text{const}.$$

Using the standard Euler–Lagrange method, we find that each solution in the two-dimensional space satisfies the equation

$$k_g = cK, \tag{1}$$

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