

## THE HARDY–LITTLEWOOD PROBLEM

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In 1922, by using extended Riemann hypothesis, Hardy and Littlewood (see [1]) found an asymptotic formula for the quantity of representations of a number in the form of the sum of a prime number and squares of two integers. Among the mathematicians contributed to solution of this problem one can refer C. Hooly (see [2]) and Yu.V. Linnik. In 1961, the last author (see [3], Chap. VII, pp. 132–180) solved this problem by the dispersion method. In [4], a solution of an indefinite analog of the Hardy–Littlewood problem was found. In [5], the asymptotics for the number of solutions of the equation  $n - x^2 - y^2 = a$  was obtained, where  $a \neq 0$  is any fixed integer,  $x, y \in \mathbb{Z}$ ,  $x^2 + y^2 < N$ ,  $N$  is a sufficiently large number,  $n \in \mathbb{N}$ ,  $n$  possesses at most 6 prime divisors, and  $p \geq N^{1/883}$  if  $p|n$ . Later, in [6], the number of representations of a natural number  $N$  in the form of the sum of a number possessing  $k$  prime divisors and two squares was found,  $2 \leq k \leq (2 - \varepsilon) \ln \ln N$  and  $(2 + \varepsilon) \ln \ln N \leq k \leq b \ln \ln N$ .

In the present article some analogs of the Hardy–Littlewood problem are solved with numbers possessing  $k$  prime divisors from arithmetic progressions definite modulo  $d_0 \leq \ln^{C_0} N$ ; besides, for all  $p$  one has  $p > \ln^{B+1} N$ .

Let  $(l_i, d_0) = 1$ ,  $p_i$  be prime numbers,  $i = 1, \dots, k$ ,  $\Omega(n)$  the quantity of prime divisors of the number  $n$  with their multiplicities taken into account. Write

$$\begin{aligned} E(t, l_1, \dots, l_k, d_0) &= \{n : n = p_1^{\alpha_1} \dots p_k^{\alpha_k}, t \leq p_1 < \dots < p_k, \\ &\quad p_i \equiv l_i \pmod{d_0}, \alpha_i \geq 0, i = 1, \dots, k, \Omega(n) = k\}, \\ &l \equiv l_1 \dots l_k \pmod{d_0}, \ln_2 x = \ln \ln x, \ln_3 x = \ln \ln \ln x. \end{aligned}$$

We shall write sums, in which the variable satisfies several conditions, in various manners: either with a series of conditions below the sign of the sum, or with a series of conditions within the square brackets neighboring the sign of the sum, i.e.,

$$\sum_{A, B} = \sum \left[ \begin{matrix} A \\ B \end{matrix} \right].$$

**Theorem 1.** Let  $\nu(N, E)$  be the quantity of representations of a natural number  $N$  in the form of the two squares of integers and a number belonging to  $E(t, l_1, \dots, l_k, d_0)$ ,  $t \geq \ln^{B+1} N$ ,  $d_0 \leq \ln^{C_0} N$ ,  $(d_0, N - l) = 1$ . Then

$$\nu(N, E) = \pi D E(N d_0) \delta(d_0) \sum_{\substack{n < N, \\ n \in E(t, l_1, \dots, l_k, d_0)}} 1 + O(N \ln^{-B+2} N + R'(N, t, k)),$$

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