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THE STABILITY OF SOLUTIONS OF NONLINEAR ALMOST PERIODIC SYSTEMS OF DELAY FUNCTIONAL-DIFFERENTIAL EQUATIONS

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For autonomous systems of differential equations $\dot{x} = f(x)$, $f(0) = 0$, the following result, strengthening the Lyapunov theorem on asymptotical stability, is known: In order for the null solution to be asymptotically stable it suffices that a positive definite function $v(x)$ exists such that $\dot{v} \leqslant 0$ and, in addition, its level surfaces $v = \text{const} > 0$ were not containing the whole trajectories (see [1], Chap. 1, § 3, p. 19). In [2] and [3], this result was extended (with intrinsic modifications in its formulation) on non-autonomous systems under assumption that the right side of the system, the Lyapunov function $v(x,t)$, and its first order partial derivatives are almost periodic with respect to t. In the present article the construction in [3] is extended on delay functional-differential equations. In contrast to the traditional schemes (see [4]–[6]), we consider only smooth Lyapunov functionals, the derivative of a functional along trajectories of the system is defined in terms of the theory of smooth mappings of Banach spaces (see [7], Chap. 1, § 2). The obtained result is new also for particular cases of autonomous and periodic systems.

Let us consider the system

$$\dot{x}(t) = f(x_t, t), \quad t \in \mathbb{R}. \tag{1}$$

Here $f(\varphi, t) : C[J] \times \mathbb{R} \to \mathbb{R}^N$, $J = [-a, 0]$, $a = \text{const} > 0$, and $C[J]$ is the Banach space of continuous functions $\varphi : J \to \mathbb{R}^N$, $x_t(\theta) = x(t+\theta)$, $\theta \in J$. We shall suppose that

1) the functional f is continuous in $C[J] \times \mathbb{R}$ and satisfies the Lipschitz condition with respect to φ;
2) f is an almost periodic functional of t uniformly with respect to φ on any closed ball in $C[J]$;
3) $f(0,t) = 0$.

A solution $x(t)$ of system (1) is uniquely determined by the initial condition $x_0(\theta) = \varphi(\theta) \in C[J]$ and represents a smooth function $\mathbb{R}_+ \to \mathbb{R}^N$, $\mathbb{R}_+ = [0, \infty)$ (see [6], Chap. 2, 2.2). In what follows we denote by $|\cdot|$ the norm in $\mathbb{R}^N$, by $\|\cdot\|$ the norm in $C[J]$, and by $C^*[J]$ the Banach space conjugate to $C[J]$. From condition 1) one can easily deduce that any solution $x(t)$ of system (1) with any $T > 0$ satisfies the following relation:

$$\sup_{t \in [0;T]} \left| \frac{x(t+\Delta t) - x(t)}{\Delta t} - f(x_t, t) \right| \to 0 \quad (\Delta t \to 0). \tag{2}$$

Let $B_r = \{\varphi \in C[J],\ \|\varphi\| \leqslant r\}$, $r > 0$, and assume that $v(\varphi, t)$ is a functional $B_r \times \mathbb{R} \to \mathbb{R}$. Its derivative with respect to φ at a point (φ, t) is a linear functional $v'_\varphi \in C^*[J]$ such that $v(\varphi + \Delta\varphi, t) = v(\varphi, t) + v'_\varphi(\Delta\varphi) + o(\|\Delta\varphi\|)$ $(\Delta\varphi \to 0)$. The functional v is said to be smooth if at any point (φ, t) it possesses derivatives v'_φ, v'_t, which continuously depend on the point (φ, t) (here

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