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Algebras Over Operad of Spheres

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Abstract—We study an operad whose components are multidimensional spheres. We give a description (up to the rational equivalence) of the variety of algebras over this operad in terms of symbols of operations and identities.

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In [1] we note that a family of multidimensional spheres forms an operad. In this paper we study this operad in detail. The main result (Theorem 5) consists in finding the explicit form (operations and identities) of a variety of algebras rationally equivalent to the variety of algebras over the operad of spheres. It turns out that operations and identities of the obtained variety bear noticeable similarities with operations and identities of the variety of convexors (see [2, 3]). Theorem 4 explains the reason of that. In addition, we adduce many examples of operads similar (in the mode of their definition) to the operad of spheres.

Multidimensional spheres are rather important objects, therefore the presence of an algebraic structure (that was not studied earlier) on the family of all multidimensional spheres represents, in our opinion, a certain interest. Note that in [1] one proves the existence of an operad structure on the family of all multidimensional (hollow) cubes centered at the origin of coordinates, whose faces are parallel to the coordinate axes, and whose edges have length 2.

Note that the operad of multidimensional spheres (as well as all other operads mentioned in this paper) give one of examples of commutative operads introduced in [4].

Here we use denotations and the terminology of [1]. See books [5–7] and paper [8] for the general theory of operads.

1. THE OPERAD OF SPHERES AND CLOSE OPERADS

The base of the further constructions is the following operad studied in [9]. Let G be a semigroup with unit 1. Let us consider the family of sets $\{G(n) \mid n \geq 1\}$, where $G(n) = G^n$. Elements of $G(n)$ are sequences (rows) $\overline{x} = (x_1, \ldots, x_n)$ of elements $x_i \in G$. The product of an element $g \in G$ and a row $\overline{x}$ is defined by the rule $g\overline{x} = (gx_1, \ldots, gx_n)$. Let us define the composition operations for all $n_1, \ldots, n_m$ as mappings in the form

$$G(m) \times G(n_1) \times \cdots \times G(n_m) \to G(n_1 + \cdots + n_m), \ \ (\overline{x}, \overline{y}_1, \ldots, \overline{y}_m) \mapsto \overline{x}\,\overline{y}_1 \ldots \overline{y}_m, \tag{1}$$

where $\overline{x} = (x_1, \ldots, x_m) \in G(m)$, $\overline{y}_i = (y_{i,1}, \ldots, y_{i,n_i}) \in G(n_i)$ for all $1 \leq i \leq m$, and $\overline{x}\,\overline{y}_1 \ldots \overline{y}_m = (x_1\overline{y}_1, \ldots, x_m\overline{y}_m)$. We assume that in the latter expression sequences $x_i\overline{y}_i$ are written without parentheses.

We also define for all n the action of a group of substitutions of the nth degree Σ_n on the set $G(n)$ by putting $\sigma(x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$ with $\sigma \in \Sigma_n$, $(x_1, \ldots, x_n) \in G(n)$.

It is known (see [9]) that the family $\{G(n) \mid n \geq 1\}$ with operations of composition and the action of groups of substitutions defined above forms an operad. For brevity we denote this operad, like the initial semigroup, by G.

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