

A NOTE ON SEMILATTICE CONGRUENCES IN ORDERED SEMIGROUPS

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For a semigroup or an ordered semigroup S we denote by $(x)_\sigma$ a σ -class S containing x ($x \in S$). If S is a semigroup and σ is a congruence on S , then the multiplication “ \cdot ” on $S/\sigma = \{(x)_\sigma \mid x \in S\}$ is defined as $(x_\sigma) \cdot (y_\sigma) = (x \cdot y)_\sigma$ for all $x, y \in S$. Then $(S/\sigma, \cdot)$ is a semigroup (see, e. g., [1]). But if (S, \cdot, \leq) is an ordered semigroup and $H \subseteq S$, then we write $[H] = \{t \in S \mid t \leq h \text{ for a certain } h \in H\}$.

For a semigroup S the following assertions are equivalent (see [2]):

- 1) a semilattice congruence σ on S exists such that $(x)_\sigma$ is a subsemigroup in S of type \mathcal{T} for each $x \in S$;
- 2) a semilattice Y and a mapping $\varphi : S \rightarrow Y$, which is a homomorphism from S to Y , exist such that $\varphi^{-1}(\{a\})$ is a subsemigroup in S of type \mathcal{T} for all $a \in Y$;
- 3) a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of subsemigroups in S of type \mathcal{T} exist such that

$$S_\alpha \cap S_\beta = \emptyset \text{ for all } \alpha, \beta \in Y, \alpha \neq \beta;$$
$$S = \bigcup_{\alpha \in Y} S_\alpha; \quad S_\alpha S_\beta \subseteq S_{\alpha\beta} \text{ for all } \alpha, \beta \in Y.$$

Our objective is to obtain a similar result for the ordered semigroups.

Let (S, \cdot, \leq) be an ordered semigroup. By *semilattice congruence* on S we call a congruence σ on S (i. e., an equivalence relation σ on S such that $(a, b) \in \sigma \Rightarrow (ac, bc) \in \sigma, (ca, cb) \in \sigma$ for all $c \in S$) such that $(a^2, a) \in \sigma, (ab, ba) \in \sigma$ for all $c \in S$ (see [3]). A semilattice congruence σ on S is said to be *complete* if $(x, y) \in \leq$ implies $(x, xy) \in \sigma$ (ibid.). A semigroup (Y, \cdot) is called a *semilattice* if $x^2 = x$ and $xy = yx$ hold for all $x, y \in Y$.

Let (Y, \cdot) be a semilattice. Let us define a relation “ \prec ” on Y as follows; $\preceq = \{(\alpha, \beta) \mid \alpha = \alpha\beta (= \beta\alpha)\}$. Then (Y, \cdot, \preceq) is an ordered semigroup.

In case of ordered groups, treating a semilattice Y , we consider it as an ordered semigroup (Y, \cdot, \preceq) with the relation \preceq , defined as follows:

$$\preceq = \{(\alpha, \beta) \mid \alpha = \alpha\beta (= \beta\alpha)\}.$$

Proposition. *Let (S, \cdot, \preceq) be an ordered semigroup. The following assertions are equivalent:*

- 1) a complete semilattice congruence σ on S exists such that $(x)_\sigma$ is a subsemigroup of type \mathcal{T} for all $x \in S$;
- 2) a semilattice Y and a homomorphism “on” $\varphi : S \rightarrow Y$ (between two ordered semigroups) exist such that $\varphi^{-1}(\{\alpha\})$ is a subsemigroup of S of type \mathcal{T} for all $\alpha \in Y$;

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