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ON THE TRIVIALITY OF NATURAL BUNDLES OF SOME COMPACT HOMOGENEOUS SPACES

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Connected compact homogeneous spaces are known to be of the form $M = G/H$, where G is a connected Lie group and H its closed subgroup. In [1], for the study of such M we introduced natural bundles. For an appropriate finite-sheeted covering space $M' = G/H'$ of M, we construct a (smooth and locally trivial) bundle $M_c \to M \to M_a$ with fiber M_c and base M_a. If G is a simply connected Lie group which acts transitively on M, and K is the maximal compact subgroup of G, then the base $M_a = K \setminus G/H'$ is an aspherical manifold (i. e., its homotopy groups $\pi_i(M_a)$ are trivial for all $i > 1$), and the fundamental group of the fiber $M_c = K/K \cap H'$ is finite. For a detailed discussion about natural bundles and their properties see [1], [2]. If a compact homogeneous space M is of the form G/Γ (where the stationary subgroup Γ is discrete and, therefore, it is a lattice in G, i. e., a discrete subgroup with compact quotient space), then for an appropriate subgroup Γ' of finite index in Γ the homogeneous space G/Γ' (which is a finite-sheeted covering space of the initial space G/Γ) possesses a natural bundle.

The present article is devoted to the study of homotopy properties of the natural bundles of compact homogeneous spaces with discrete stationary subgroups and their applications to the investigation of arbitrary compact homogeneous spaces.

If the fundamental group $\pi_1(G)$ of a Lie group G is finite and K is a maximal compact subgroup in G (arbitrarily chosen, because all such subgroups are conjugate in G), then the natural bundle is of the form $M_c \to G/\Gamma' = M' \to K \setminus G/\Gamma' = M_a$. It can be assumed that $\Gamma' \cap K$ lies in the center $Z(G)$ of G and that the quotient group $\Gamma'/\Gamma' \cap K$ is torsion-free (see [1]). Let $\widehat{G} = G/\Gamma' \cap K, \widehat{\Gamma} = \Gamma'/\Gamma' \cap K, \widehat{K} = K/K \cap \Gamma'$. Here $\widehat{K}$ is the maximal compact subgroup of the Lie group $\widehat{G}$, $\widehat{\Gamma}$ a lattice in $\widehat{G}$, $\widehat{G}/\widehat{\Gamma}$ is diffeomorphic to $M' = G/\Gamma'$, and $\widehat{K} \to \widehat{G}/\widehat{\Gamma} = M' \to M_a$ is the principal $\widehat{K}$-bundle over M_a which is a natural bundle for the homogeneous manifold M'. In addition, $\pi_1(\widehat{G})$ is a finite group, and the lattice $\widehat{\Gamma}$ is torsion-free. Therefore, if a homogeneous manifold $M = G/\Gamma$ is studied up to a finite-sheeted covering, the conditions imposed on G and Γ in the following theorem are not exaggeratedly restrictive. The understanding of the statement of this theorem requires a certain acquaintance with the classification of the semisimple real Lie algebras (the necessary information, in particular, the list of all Lie algebras of the category I, can be found, e. g., in [3]).

Theorem 1. *Let G be a connected Lie group with finite fundamental group and without compact semisimple factors, and Γ a torsion-free lattice in G. Then*

(i) *if the Lie algebra $L(S)$ of the semisimple part S (i. e., the Levi factor) of G possesses at least one ideal which is either of the category I or isomorphic to one of the following Lie algebras: $sl(2m, \mathbf{R})$, $so(2k+1, 2l+1)$, or E_6I, and the center $Z(S)$ is finite, then $C(G, \Gamma, \mathbf{R}) \neq \{0\}$;*

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