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ARITHMETICAL EMBEDDINGS OF SIMPLE CENTRAL ALGEBRAS INTO MATRIX ALGEBRAS

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In this article we generalize results of [1] to the embeddings of simple central algebras into matrix algebras. In his works on quaternion arithmetics, B.A. Venkov introduced (by analogy with the complex numbers) concepts of the rotations of quaternions. Developing the theory of rotations, Yu.V. Linnik applied this theory to the study of representations of numbers by ternary quadratic forms. Further, the other authors developed the rotation theory generalizing it to orders in algebras of generalized quaternions. A detailed bibliography on this theme can be found in [2]. In the same work, instead of rotation of quaternions the authors suggested to consider "rotations" of embeddings satisfying a certain "arithmetical" condition, which allowed them to generalize the theory of rotations to simple central algebras over algebraic number fields.

Let $\mathfrak{A}$ be an n-dimensional simple central algebra over the field of rational numbers $\mathbb{Q}$. As usual, we use the subscript A to denote adele objects corresponding to the global ones: $\mathfrak{A}_A$ is the adele ring of an algebra $\mathfrak{A}$, $\mathbb{Q}_A$ the adele ring of the field $\mathbb{Q}$, etc. By $\mathfrak{A}^*_A$, $\mathbb{Q}^*_A$, $\mathfrak{O}^*_A$ and so on we will denote the ideles (i. e., invertible adeles) of the corresponding adele rings. If $\lambda_A \in \mathfrak{A}^*_A$, then $\lambda_A = (\dots, \lambda_p, \dots)$, where λ_p stands for the p-component and $\|\lambda_p\|_p = 1$ for almost all p running through the prime numbers in $\mathbb{Z}$ and ∞. A more detailed description of the adele objects can be found in [3] (p. 271).

Let $M_n(\mathbb{Q})$ be the algebra of $n \times n$ matrices over $\mathbb{Q}$. By the Skolem–Noether theorem, all embeddings of an algebra $\mathfrak{A}$ into $M_n(\mathbb{Q})$ are conjugate and therefore indistinguishable from the algebraic point of view. Let us consider these embeddings from the arithmetical point of view. Let $\mathfrak{O}$ be an order in $\mathfrak{A}$. Consider the set of embeddings τ of $\mathfrak{A}$ into $M_n(\mathbb{Q})$ satisfying the following arithmetical condition

$$\tau(\mathfrak{O}) = \tau(\mathfrak{A}) \cap M_n(\mathbb{Z}). \tag{1}$$

If V is an arbitrary matrix in $GL_n(\mathbb{Q})$, then $V^{-1}\tau V$ does not need to satisfy condition (1). At the same time, if $V \in GL_n(\mathbb{Z})$ (i. e., V is an integer matrix and $\det V = \pm 1$), then τ and $V^{-1}\tau V$ satisfy condition (1) simultaneously.

Definition 1. Two embeddings τ_1 and τ_2 of $\mathfrak{A}$ into $M_n(\mathbb{Q})$ are said to be equivalent if a matrix V in $GL_n(\mathbb{Z})$ exists such that $\tau_2 = V^{-1}\tau_1 V$.

Now let I be an ideal in $\mathfrak{A}$ and $\mathfrak{O}_I$ its left order. Let us fix a basis $\omega_1, \dots, \omega_n$ in I and consider the embedding τ_I of $\mathfrak{A}$ into $M_n(\mathbb{Q})$ defined as follows:

$$\tau_i(\alpha) \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix} = \alpha \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix}.$$