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A Problem on Brachistochrone as Invariant Variational Problem

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Abstract—Based on the modified theory of invariant variational problems developed by the author, we describe a theoretical group approach to the problem of brachistochrone.

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This problem was set by Johann Bernoulli (in 1696), one of the founders of Calculus of variations, and soon was solved by him in 1697 (this solution was published in Acta Eruditorum, 1697). Afterwards, it was also solved by I. Newton, G. Leibniz, Jakob Bernoulli and G. de l'Hôpital (see, e.g., [1]). Let us recall this problem.

Given two points A and B lying in a vertical plane but not on the same vertical line. What is the trajectory of a point moving by the gravity force only, starting from point A and reaching point B in the shortest time?

This curve happens to be a cycloid, and it has parametric equations in the form

$$\begin{aligned} x &= a(t - \sin t), \\ y &= a(1 - \cos t), \end{aligned} \qquad t \in [0, 2\pi]. \tag{1}$$

Here a is a radius of rolling circle, and t is a rotation angle.

In Calculus of variations, this problem is reduced to finding the minimum of the functional

$$T[y(x)] = \frac{1}{\sqrt{2g}} \int_0^l \sqrt{\frac{1 + y'^2}{y}} dx. \tag{2}$$

Equalities (1) are Euler equations that provide necessary minimum conditions for functional (2). Later on, it was shown that brachistochrone actually gives strong minimum (e.g., [2]).

We will show that all known results (first integral, time of the steepest descent) are attainable from the unified ground of modified theory of invariant variational problems [3].

Consider the integral $I = \int\limits_{D_0} L\big(x, \varphi, \frac{\partial \varphi}{\partial x}\big) dx$, where $x = (x^1, \ldots, x^n)$ are independent variables, $\varphi = (\varphi^1, \ldots, \varphi^m)$ are dependent ones. Denote by $\frac{\partial \varphi}{\partial x}$ a complex of the first order partial derivatives

$$\varphi_i^k = \frac{\partial \varphi^k}{\partial x^i}, \;\; i = 1, \ldots, n, \;\; k = 1, \ldots, m;$$

and Lagrangian L is a smooth function.

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